What is a vertex algebra?

Chenjing Bu

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Before explaining what a vertex algebra is, it might be useful to first introduce a geometric way of thinking about associative algebras.

Namely, we can think of an associative algebra as a one-dimensional structure, as living on a line $\mathbb{R}$, in the sense that given several points on the line, if we assign an element of the algebra to each point, we can multiply them in the order given by the positions of the points on the line. The product does not change when we move the points around, as long as they do not cross over each other. This is illustrated on the left of Figure 1.

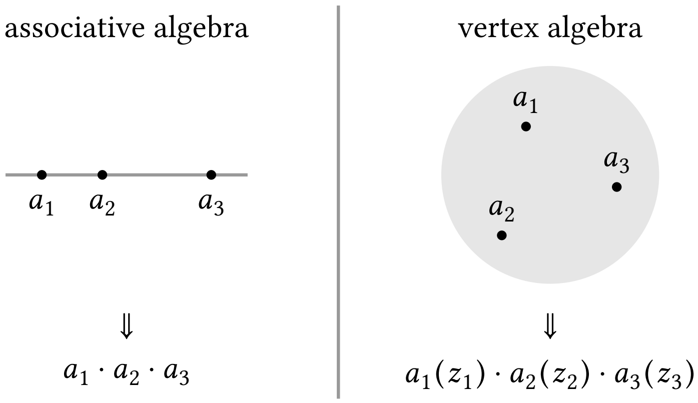

Figure 1. Associative algebras and vertex algebras.

In comparison, a vertex algebra is a two-dimensional structure. In a vertex algebra, we multiply elements on the complex plane $\mathbb{C}$. Morally speaking, given several points $z_1, \ldots, z_n \in \mathbb{C}$, if we assign an element a_i to each point z_i, we get a product $a_1(z_1) \cdots a_n(z_n)$. This product no longer stays fixed when we move the points around – otherwise we would have defined a usual commutative algebra – but it depends meromorphically on the variables z_i, and can only have poles when $z_i = z_j$ for some $i \neq j$, that is, when two of the points collide.

More precisely, a vertex algebra can be defined as follows:

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Definition 1. A *vertex algebra* over $\mathbb{C}$ is the following data:

- a $\mathbb{C}$-vector space V;
- for each integer $n \geq 0$, a multiplication map

$$\begin{aligned} V^{\otimes n} &\longrightarrow V[\![z_1, \ldots, z_n]\!]\,[(z_i - z_j)^{-1}]\,, \\ a_1 \otimes \cdots \otimes a_n &\longmapsto a_1(z_1) \cdots a_n(z_n)\,, \end{aligned}$$

  where $V[\![z_1, \ldots, z_n]\!]$ is the space of formal power series in n variables valued in V, and we invert $z_i - z_j$ when $i \neq j$.

The *unit element* $1 \in V$ is defined to be the result of the 0-ary multiplication. This data needs to satisfy the following axioms:

- For any $a \in V$, we have $a(z) \in a + z\,V[\![z]\!]$, or more concisely,

$$a(0) = a\,.$$

- (*Commutativity*) For any $a_1, \ldots, a_n \in V$ and any permutation σ of $\{1, \ldots, n\}$, we have

$$a_{\sigma(1)}(z_{\sigma(1)}) \cdots a_{\sigma(n)}(z_{\sigma(n)}) = a_1(z_1) \cdots a_n(z_n)$$

  in $V[\![z_1, \ldots, z_n]\!]\,[(z_i - z_j)^{-1}]$.
- (*Associativity*) For any $a_1, \ldots, a_n, b_1, \ldots, b_m \in V$, we have

$$\begin{aligned} &\big[b_1(w_1) \cdots b_m(w_m)\big](z_0)\, a_1(z_1) \cdots a_n(z_n) \\ &\qquad = b_1(z_0 + w_1) \cdots b_m(z_0 + w_m)\, a_1(z_1) \cdots a_n(z_n) \end{aligned}$$

  in the ring

$$V[\![z_0, \ldots, z_n]\!]\,[(z_i - z_j)^{-1}]\,[\![w_1, \ldots, w_m]\!]\,[(w_i - w_j)^{-1}]\,.$$

  Here, the left-hand side of the equation naturally lies in this ring; the right-hand side is re-expanded using the formula

$$\frac{1}{z_0 + w_i - z_j} = \sum_{n=0}^{\infty} \frac{(-1)^n\, w_i^n}{(z_0 - z_j)^{n+1}}\,,$$

  that is, we expand in non-negative powers of w_i.

The associativity axiom has a straightforward geometric interpretation, as shown in Figure 2: if we take the product of the elements b_i placed at w_i, then translate everything along the vector z_0, and then take the product together with the elements a_j placed at z_j, it is the same

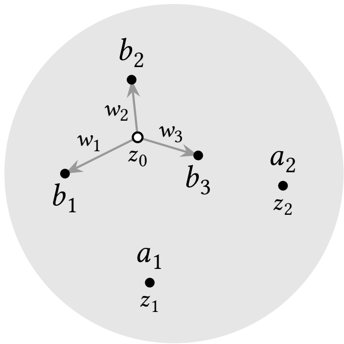

Figure 2. An example of the associativity axiom.

as taking the product of all the elements b_i and a_j at once, at their respective positions $z_0 + w_i$ and z_j.

The reader should be careful that the notation $a_1(z_1) \cdots a_n(z_n)$ here means applying the multiplication map to the elements $a_1, \ldots, a_n$, and it is not meant to be an operation on the individual terms $a_i(z_i)$, meaning the result of the 1-ary multiplication, which appeared in the first axiom.

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The above is, however, not the conventional way to define a vertex algebra. A more common definition, such as those in [6, 10] (with slight variations), goes as follows:

Definition 2. A *vertex algebra* over $\mathbb{C}$ is the following data:

- a $\mathbb{C}$-vector space V;
- a unit element $1 \in V$, also called the *vacuum*;
- an operator $T\colon V \to V$, called the *translation operator*;
- a multiplication map

$$\begin{aligned} Y\colon V \otimes V &\longrightarrow V[\![z]\!]\,[z^{-1}]\,, \\ a \otimes b &\longmapsto Y(a, z)\, b\,. \end{aligned}$$

For each $a \in V$, we regard $Y(a, z)$ as an operator-valued formal power series in z, that is, as an element of $\operatorname{End}(V)\,[\![z, z^{-1}]\!]$. We require that this data satisfies the following axioms:

- We have $Y(1, z) = \mathrm{id}_V$, and for any $a \in V$, we have $Y(a, z)\, 1 \in a + z\, V[\![z]\!]$.
- (*Translation*) We have $T(1) = 0$, and for any $a \in V$, we have the commutator

$$[T, Y(a, z)] = \frac{\partial}{\partial z}\, Y(a, z)\,.$$

- (*Locality*) For any $a, b \in V$, there exists an integer $n > 0$ such that

$$(z - w)^n \cdot [Y(a, z), Y(b, w)] = 0$$

in $\mathrm{End}(V)\, [\![z^{\pm 1}, w^{\pm 1}]\!]$.

This definition may look overwhelming at a first glance, but it turns out to be equivalent to Definition 1. The multiplication map Y here, in the previous language, corresponds to the multiplication

$$Y(a, z)\, b = a(z)\, b(0)\ ,$$

and the translation operator T here is given by

$$T a = \frac{\partial}{\partial z}\, a(z) \Big|_{z=0}\ ,$$

where $a(z)$ here denotes the 1-ary multiplication map in Definition 1. The name 'translation operator' comes from the stronger property

$$(T a)(z) = \frac{\partial}{\partial z}\, a(z)\ ,$$

which can be deduced from axioms in Definition 1, and is also the special case of the translation axiom in Definition 2 when applied to $1 \in V$; it implies that

$$(\exp(zT)\, a)(w) = \exp\Big(z\, \frac{\partial}{\partial w} \Big)\, a(w) = a(z + w)\ ,$$

where the second step is a formal fact. In other words, the operator $\exp(zT)$ is, in this sense, the actual translation operator which translates $a(w)$ by z. In particular, taking $w = 0$ gives the identity

$$\exp(zT)\, a = a(z)\ ,$$

meaning roughly that $\exp(zT)$ acts on an element a by 'placing a at z'.

The locality axiom here is a convoluted way of expressing the commutativity and associativity axioms at the same time, by requiring that for any $a, b, c \in V$, we have

$$a(z)\, b(w)\, c(0) = b(w)\, a(z)\, c(0)\ .$$

The extra factor $(z - w)^n$ is inserted because in the statement of the locality axiom, rational functions are expanded into power series in different orders. For example, we have

$$-\sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}} \sim \frac{1}{z - w} \sim \sum_{n=0}^{\infty} \frac{w^n}{z^{n+1}}\ ,$$

where the two sides are results of expanding the rational function $1/(z - w)$ in non-negative powers of z and w, respectively. However, the difference of the two sides is a zero-divisor in the ring $\mathbb{C}[\![z^{\pm 1}, w^{\pm 1}]\!]$, and is annihilated by $z - w$:

$$(z - w) \cdot \left(\sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}} + \sum_{n=0}^{\infty} \frac{w^n}{z^{n+1}} \right) = (z - w) \cdot \sum_{n=-\infty}^{\infty} w^n \, z^{-1-n} = 0 \, .$$

The locality axiom can therefore be understood as requiring that for any $c \in V$, we have $Y(a, z) \, Y(b, w) \, c = Y(b, w) \, Y(a, z) \, c$ as rational functions in z and w, and moreover, that this rational function is only allowed to have poles at $z = 0$, $w = 0$, and $z = w$.

Conversely, the n-ary multiplication map in Definition 1 can be recovered from the standard multiplication map Y as

$$a_1(z_1) \cdots a_n(z_n) = Y(a_1, z_1) \cdots Y(a_n, z_n) \, 1 \, .$$

The equivalence of the two definitions essentially follows from [11, Theorem 3.14], and a proof can be found in [3, §2.1.5].

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The above two approaches to understanding vertex algebras have their respective advantages and disadvantages, which we illustrate with some examples.

On one hand, from a purely expository point of view, the n-ary multiplication in Definition 1 can often be more intuitive. For example, the straightforward relation

$$a(z) \, b(0) = b(0) \, a(z) \, ,$$

is sometimes encoded as

$$Y(a, z) \, b = \exp(zT) \, Y(b, -z) \, a$$

in the standard notation of Definition 2. Namely, the right-hand side can be decoded into $\big(b(-z) \, a(0)\big)(z) = b(0) \, a(z)$, meaning that translating everything in $b(-z) \, a(0)$ by z gives $b(0) \, a(z)$.

Also, Definition 1 often makes it easier to define generalized notions of vertex algebras that, for example, are allowed to have more types of poles, or are allowed to be non-commutative. This is because these properties are expressed directly in the axioms in this approach, rather than through a less direct locality axiom.

On the other hand, the operators $Y(a, z)$ from Definition 2 are crucial in understanding the structure of vertex algebras in many ways. For example, they are used to define *normally ordered products* of operators, which we explain now.

Suppose that we are given operator-valued formal power series $A_1(z), \ldots, A_k(z)$, where $A_i(z) = \sum_{n\in\mathbb{Z}} A_{i,n}\, z^{-n-1}$ with $A_{i,n} \in \operatorname{End}(V)$. Here, we are using capital letters for the operators, to distinguish them from the notation for n-ary multiplication. For example, we may take $A_i(z) = Y(a_i, z)$ for elements $a_i \in V$.

The *normally ordered product* of the operators $A_1(z_1), \ldots, A_k(z_k)$ is then defined by

$$:A_1(z_1) \cdots A_k(z_k): \quad = \sum_{\substack{\{1,\ldots,k\}= \\ \{i_1<\cdots<i_p\}\sqcup\{j_1<\cdots<j_q\}}} A_{i_1}(z_{i_1})_+ \cdots A_{i_p}(z_{i_p})_+\, A_{j_q}(z_{j_q})_- \cdots A_{j_1}(z_{j_1})_- \,,$$

where

$$A_i(z)_+ = \sum_{n<0} A_{i,n}\, z^{-n-1}\,, \qquad A_i(z)_- = \sum_{n\geq 0} A_{i,n}\, z^{-n-1}$$

are the parts of $A_i(z)$ consisting of its non-negative and negative *modes* (meaning coefficients of powers of z), so that $A_i(z) = A_i(z)_+ + A_i(z)_-$. For example, when $k = 2$, we have

$$:A(z)\, B(w): \ = \ A(z)_+\, B(w) \ + \ B(w)\, A(z)_- \,.$$

The idea of doing such a reordering is to avoid singularities along $z = w$. Namely, in a vertex algebra, we have the *operator product expansion* (*OPE*), which refers to the identity

$$\begin{aligned} Y(a, z)\, Y(b, w) &= \sum_{n\in\mathbb{Z}} \frac{Y(a_n\, b, w)}{(z-w)^{n+1}} \\ &= \sum_{n\geq 0} \frac{Y(a_n\, b, w)}{(z-w)^{n+1}} + :Y(a, z)\, Y(b, w):\,, \end{aligned}$$

where $a_n\, b \in V$ denotes the coefficient of z^{-n-1} in $Y(a, z)\, b = a(z)\, b(0)$. Note that the operator product $Y(a, z)\, Y(b, w)$ acts on a third element, so it describes multiplying 3 elements in the vertex algebra. The first equality above follows directly from the associativity property

$$a(z)\, b(w)\, c(0) = [a(z-w)\, b(0)](w)\, c(0)\,,$$

whereas the second one is less obvious, but still standard.

This exhibits the normally ordered product $:Y(a, z)\, Y(b, w):$ precisely as the non-singular part of the product $Y(a, z)\, Y(b, w)$ along the diagonal $z = w$. In particular, we can now restrict it to the diagonal, giving a normally ordered product $:Y(a, z)\, Y(b, z):$.

4

A basic example of a vertex algebra is the *Heisenberg vertex algebra*, a special case of the *affine vertex algebra* associated to the affine Lie algebra $\widehat{\mathfrak{gl}}_1$. The latter is defined as a vector space by

$$\widehat{\mathfrak{gl}}_1 = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \cdot b_n \oplus \mathbb{C} \cdot K ,$$

and the Lie bracket is given by

$$\begin{aligned} [b_n, b_m] &= n\, \delta_{n+m,0} \cdot K , \\ [K, -] &= 0 . \end{aligned}$$

It is a central extension of the *loop Lie algebra* $L(\mathfrak{gl}_1) = \mathfrak{gl}_1[t, t^{-1}]$, which is an abelian Lie algebra, with central element K.

The underlying vector space V of the Heisenberg vertex algebra is a $\widehat{\mathfrak{gl}}_1$-representation,

$$\begin{aligned} V &= U(\widehat{\mathfrak{gl}}_1) \cdot 1 \Big/ \left\{ \begin{matrix} b_n \cdot 1 = 0 & (n \geq 0) \\ K \cdot 1 = k \cdot 1 & \end{matrix} \right\} \\ &\simeq \mathbb{C}[b_{-1}, b_{-2}, \ldots] , \end{aligned}$$

where $U(-)$ denotes the universal enveloping algebra, and we choose a *level* $k \in \mathbb{C}$. The second line is a PBW isomorphism, and is only a non-canonical isomorphism of vector spaces. The generator $1 \in V$ will become the unit element of the vertex algebra.

The Heisenberg vertex algebra can be seen as generated by a single field (meaning an operator-valued formal power series),

$$B(z) = \sum_{n \in \mathbb{Z}} b_n \, z^{-n-1} \in \operatorname{End}(V) [\![z, z^{-1}]\!] ,$$

where each b_n acts on V via the $\widehat{\mathfrak{gl}}_1$-action.

The vertex algebra product is determined by the formula

$$Y(b_{-n-1}, z) = \frac{1}{n!} \left(\frac{\partial}{\partial z} \right)^n B(z)$$

for $n \geq 0$, where b_{-n-1} denotes the element $b_{-n-1} \cdot 1 \in V$. More general vertex products in V are also determined by this formula, since it can be shown that we must have

$$Y(b_{-n_1-1} \cdots b_{-n_m-1}, z) = {:}Y(b_{-n_1-1}, z) \cdots Y(b_{-n_m-1}, z){:} ,$$

where again, $b_{-n_1-1} \cdots b_{-n_m-1}$ refers to the element $b_{-n_1-1} \cdots b_{-n_m-1} \cdot 1 \in V$, and $:\cdots:$ denotes the normally ordered product.

The operator product expansion for the Heisenberg vertex algebra is given on the generating field $B(z)$ by

$$B(z)\, B(w) \sim \frac{k}{(z-w)^2}\,,$$

where $\sim$ means that we omit the regular part $:B(z)\, B(w):$ from the right-hand side, which is a customary notation, and k is the level.

The Heisenberg vertex algebra describes the local behaviour of a conformal field theory, namely that of a free boson. The Lie algebra $L(\mathfrak{gl}_1)$ describes the infinitesimal symmetries of this theory, and the central extension is a general phenomenon in quantum field theory, where we usually only get projective representations of a Lie algebra, or representations up to scaling, which correspond to representations of a central extension.

5

The concept of vertex algebras is usually associated with that of two-dimensional conformal field theories, since the latter are a main source of vertex algebras. However, vertex algebras can also appear in other unexpected situations.

A recent construction of Joyce ([8, 9]; see also [3, 7, 12]) gives a vertex algebra structure on the homology of certain moduli spaces, such as moduli spaces of quiver representations, or moduli spaces of coherent sheaves on complex smooth varieties.

This type of moduli spaces have long been known to produce rich algebraic structures, including various types of quantum groups; see [2, 13, 14, 16, 17] for some examples. On the other hand, these new vertex algebras are rather mysterious, as they do not seem to come from a known conformal field theory. Instead, they are motivated by enumerative geometry. When crossing a wall in the space of stability conditions, Joyce [9] expressed the change in enumerative invariants, or intersection pairings on moduli spaces, in terms of products in this vertex algebra.

The simplest interesting example of such a vertex algebra comes from the homology of the classifying space of $\mathrm{GL}(n;\mathbb{C})$, which is

$$\mathrm{H}_\bullet(\mathrm{BGL}(n;\mathbb{C});\mathbb{C}) \simeq \mathbb{C}[t_1, \ldots, t_n]\,,$$

where t_i has degree $2i$, and is dual to the i-th universal Chern character $\mathrm{ch}_i \in \mathrm{H}^{2i}(\mathrm{BGL}(n;\mathbb{C});\mathbb{C})$. Letting $n \to \infty$ gives

$$\mathrm{H}_\bullet(\mathrm{BGL}(\infty;\mathbb{C});\mathbb{C}) \simeq \mathbb{C}[t_1, t_2, \ldots]\,,$$

where $\mathrm{GL}(\infty; \mathbb{C})$ is defined as a colimit of the groups $\mathrm{GL}(n; \mathbb{C})$. Its classifying space $\mathrm{BGL}(\infty; \mathbb{C})$ has an interpretation as the moduli space of chain complexes of $\mathbb{C}$-vector spaces (of a fixed Euler characteristic).

This homology, as a vector space, coincides with the underlying space of the Heisenberg vertex algebra. Indeed, Joyce's construction in this case does give the Heisenberg vertex algebra. See [12] for more explanations and more examples.

This is not the only type of vertex algebra that appears in this setup. There is another type of vertex algebras, perhaps more motivated physically, which are related to the *AGT conjecture* [1], a version of which was proved in [15]; see also [4, 5, 13] for related constructions and conjectures.

It seems that the two types of vertex algebras are different but related, and finding a precise way of relating them is a very interesting open question.

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bucj@mailbox.org

Mathematical Institute, University of Oxford, Oxford OX2 6GG, United Kingdom.