Stacks and combinatorics in enumerative geometry

Chenjing Bu

Abstract. This is an introductory survey on a new intrinsic framework of enumerative geometry, which allows certain constructions of invariants to be interpreted intrinsically to the geometry of the moduli stack, and consequently, to be extended to much more general stacks than previously possible. Such constructions also allow us to prove interesting general properties of algebraic stacks. We explain several results in this direction, including decomposition theorems in cohomology and for derived categories of coherent sheaves, and discuss some potential applications that they open up.

1. Overview

1.1. Background

Enumerative geometry, in its modern form, is the study of various moduli spaces and their geometry, and in particular, of invariants extracted from them, called *enumerative invariants*. Such invariants can be numbers, vector spaces, categories, or other structures. For a moduli space M, typical invariants that are of interest include the following:

- the Euler characteristic $\chi(M)$;
- the cohomology $\mathrm{H}^\bullet(M)$, as a vector space, or as a Hodge structure;
- intersection pairings on $\mathrm{H}^\bullet(M)$, or equivalently, the fundamental class $[M] \in \mathrm{H}_\bullet(M)$;
- the derived category of coherent sheaves $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(M)$;
- etc.

These all make sense when M is a smooth projective complex variety, but moduli spaces in reality are usually not as nice. There are three main obstacles to constructing these invariants:

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1. M can be non-compact, and might not have a well-behaved intersection theory. This usually requires one to construct a compactification of M, and this process is specific to each moduli problem. For example, this involves considering singular curves in addition to smooth ones in Gromov–Witten theory, and considering coherent sheaves in addition to vector bundles in sheaf-counting theories.
2. M can be singular, which also affects its intersection theory. A general philosophy of dealing with this has now been well-developed, that is to use the *virtual*, or *derived*, geometry on M, instead of classical geometry, and many interesting singular moduli spaces have some sort of smoothness hidden in the derived structure.
3. Points in M can have non-trivial automorphisms, and as a consequence, M is more naturally seen as a *stack*, rather than a space. This is comparable to the situation when we consider a collection of mathematical objects that have non-trivial automorphisms, it is usually more natural to consider this collection as a category, rather than a set.

Our main focus is to address the problem (3), which has arguably been the least understood of the three until recently. It also affects more types invariants listed above, since for example, the Euler characteristic and the cohomology are still reasonably well-behaved even if M is a non-compact singular complex variety, but if M is a genuine (Artin) stack, these often fail to give finite invariants.

As a simplest example, consider the classifying stack $*/\mathbb{C}^\times$, which has a single point whose automorphism group is $\mathbb{C}^\times$. The reader unfamiliar with stacks may think of this as the topological classifying space $\mathrm{B}\mathbb{C}^\times \simeq \mathbb{CP}^\infty$. Its Euler characteristic is $\chi(*/\mathbb{C}^\times) = 1/0 = \infty$, and its cohomology $\mathrm{H}^\bullet(*/\mathbb{C}^\times; \mathbb{Q}) \simeq \mathbb{Q}[t]$ is infinite-dimensional, which do not give finite invariants that we would like to obtain.

1.2. Linear moduli stacks

Nevertheless, there is a class of Artin stacks for which it is known how to solve the problem mentioned above, which are moduli stacks of objects in linear categories, or *linear moduli stacks* in the sense of [20, §7.1].

Many different flavours of enumerative invariants have been developed in the linear case in the past decades. Notable examples include the following:

1. Intersection pairings on moduli spaces of semistable vector bundles on curves. These were computed by Jeffrey–Kirwan [40] when the moduli space is not stacky, that is when the rank and the degree are coprime, and later extended to the non-coprime case in [15, 23, 39].
2. Invariants counting coherent sheaves on algebraic surfaces, sometimes called *algebraic Donaldson invariants*, studied by Mochizuki [58] and Joyce [47].

3. *Donaldson–Thomas invariants* counting coherent sheaves on Calabi–Yau threefolds, or counting representations of quivers with potentials. This was initiated by Donaldson–Thomas [28] and Thomas [77], and later developed in the stacky case by Joyce [42, 43, 44, 45, 46], Joyce–Song [48], and Kontsevich–Soibelman [52].
4. *Cohomological Donaldson–Thomas theory*, categorifying Donaldson–Thomas invariants by producing vector spaces whose dimensions give the numerical invariants. Notable foundational works include Kontsevich–Soibelman [53], Efimov [31], Meinhardt–Reineke [57], and Davison–Meinhardt [26].
5. *Categorical Donaldson–Thomas theory*, further categorifying cohomological Donaldson–Thomas theory by producing invariants which are categories. This theory was developed by Toda [79] and studied in a series of examples by Pădurariu–Toda [60, 61, 62, 63, 64, 66, 67].
6. *Vafa–Witten invariants* counting Higgs sheaves on surfaces, due to Vafa–Witten [81] and developed in the works of Tanaka–Thomas [75, 76] and Thomas [78].

However, in most cases, the constructions of such invariants involve techniques that only work in the linear case, and although it has been expected that many of these invariants can be extended to non-linear cases, it has not been clear how to do this until recently.

1.3. Intrinsic enumerative geometry

The programme of *intrinsic enumerative geometry*, being developed in a series of works [19, 20, 21, 22, 24, 36], is aimed at solving this problem, and builds upon the *intrinsic moduli theory* developed in the works [3, 4, 5, 6, 35] by Alper, Hall, Halpern-Leistner, Heinloth, and Rydh. It has two main goals:

- Interpret all the invariants mentioned in §1.2, and potentially more, as intrinsic to the geometry of the moduli stacks in question, without relying on auxiliary data provided by a linear category.
- By doing this, extend these invariants to more general stacks, such as moduli stacks of G-bundles or G-Higgs bundles on curves, for a reductive group G that is not necessarily GL_n, SL_n, or PGL_n.

In the series of work mentioned above, we have already been able to achieve both goals for the enumerative theories (3)–(5) (and partially (6)) in §1.2, and we will discuss them in detail in §3 below.

Even in the linear case, by taking an intrinsic point of view, we have sometimes been able to remove certain technical restrictions from known results, thus making them applicable to a wider range of examples. This is the case, for example, in [19, 24].

Apart from being able to construct enumerative invariants, this programme also has the following motivations and potential applications:

- It provides a better understanding of the structure of the cohomology of stacks and the category of coherent sheaves on stacks. Namely, as we mentioned, the cohomology of a stack is usually infinite-dimensional, but the decomposition theorem from [19, 36] describes this infinite-dimensional space using a finite amount of data. Similarly, in [24], we also decompose the category of coherent sheaves on a stack into pieces that are easier to understand. See §§3.2–3.3 for more details on these results.
- The *geometric Langlands programme.* A surprising link between Donaldson–Thomas theory and the Dolbeault geometric Langlands conjecture was observed by Pădurariu–Toda [69], where they use semiorthogonal decompositions from Donaldson–Thomas theory to give a first precise formulation of this conjecture. In [24], using intrinsic enumerative geometry, we extend their idea from type A groups to all reductive groups, and we also formulate a conjectural refined version of the de Rham geometric Langlands correspondence proved in [7, 8, 25, 33, 34]. See §3.3 for more details.
- *S-duality.* Generating series of Vafa–Witten invariants have modular properties as a consequence of S-duality, first proposed by Vafa–Witten [81]; see for example [1, 2]. However, the mathematical definition of Vafa–Witten invariants is not known for groups that are not of type A, except a motivic version for type B/C/D groups [17] defined using the intrinsic framework. It is an interesting problem to establish modular properties and Langlands duality for Vafa–Witten invariants outside of type A.
- *Geometric representation theory.* Various types of Hall algebras (finite, cohomological, K-theoretic, categorical, etc.) are algebraic structures constructed from moduli stacks, closely tied to enumerative geometry, and often carry rich structures by themselves, such as those of quantum groups and vertex (co)algebras [14, 41, 47, 54, 55, 56, 60, 61, 71, 72, 73, 82]. The intrinsic framework should help understand these structures better, and it is interesting to explore similar structures outside of the linear case. See [18, 27] for some results in the orthosymplectic setting.

1.4. An open question

So far, we have not yet addressed the invariants (1)–(2) in §1.2, which are generalized intersection pairings on moduli stacks. These seem more difficult to generalize, because in the linear case, such invariants are constructed using auxiliary moduli spaces, such as those of *Bradlow pairs* as in [13, 47, 48, 58], which are no longer stacky, but their natural generalizations to non-linear moduli stacks can still be stacky, and cannot be used directly to construct invariants.

However, there are possible ways around this problem, which we hope to explore in the future. One of them is to use *Kirwan blow-ups* [30, 51] to eliminate stacky-ness by a canonical sequence of blow-ups, and some progress has been made by Kiem–Li–Savvas [49].

2. The combinatorics of stacks

A central new concept in intrinsic enumerative geometry is the study of the combinatorics of stacks, encoded in the *component lattice* of a stack, introduced in [20]. It can be seen as a generalization of root systems from Lie groups to stacks. We explain this construction in this section.

2.1. Lie groups and root systems

Let G be a connected complex reductive Lie group, such as GL_n, SL_n, SO_n, Sp_{2n}, etc.

Let $T \subset G$ be a maximal torus, so that $T \simeq (\mathbb{C}^\times)^r$, where r is called the *rank* of G. Consider the *cocharacter lattice* and the *character lattice* of T,

$$\Lambda_T = \mathrm{Hom}(\mathbb{C}^\times, T) \ , \qquad \Lambda^T = \mathrm{Hom}(T, \mathbb{C}^\times) \ .$$

These are isomorphic to $\mathbb{Z}^r$, and are dual to each other.

The *Weyl group* $W = \mathrm{N}_G(T)/T$ acts on both lattices, acting freely and transitively on the set $\Phi \subset \Lambda^T$ of *roots* of G. Each root defines a hyperplane in $\Lambda_T \otimes \mathbb{R}$, forming a hyperplane arrangement, and W acts freely and transitively on the set of open chambers in $\Lambda_T \otimes \mathbb{R}$.

Definition 2.1. The *component lattice* of the group G, or more precisely, of the classifying stack $*/G$, as a set, is the quotient

$$\mathrm{CL}(*/G) = \Lambda_T/W \ .$$

Extra structures on this set will be introduced later.

One may think of this as a single chamber in Λ_T, but it will turn out also to be helpful to think of it as a stacky, or orbifoldy, quotient.

Many facts in Lie theory can be formulated using the component lattice:

1. *Cocharacters*: Λ_T/W is naturally identified with the set of cocharacters of G, i.e. homomorphisms $\lambda\colon \mathbb{C}^\times \to G$, up to conjugation in G. Indeed, any cocharacter is conjugate to one that lands in T, and two cocharacters of T are conjugate if and only if they are in the same W-orbit.

2. *Cohomology*: It is well-known that the cohomology of the classifying space $\mathrm{B}G$ of G, which is the same as the cohomology of the classifying stack $*/G$, is

$$\mathrm{H}^\bullet(*/G;\mathbb{Q}) \simeq \mathbb{Q}[t_1,\dots,t_r]^W \ ,$$

the space of W-invariant polynomial functions on Λ_T, or in other words, the space of polynomial functions on the quotient Λ_T/W.

3. *Representations*: It is also well-known that irreducible representations of G are in one-to-one correspondence with *dominant* characters of T, which form a chamber $\Lambda^T_+ \subset \Lambda^T$. The category $\mathsf{Rep}(G)$ of finite-dimensional G-representations over $\mathbb{C}$ is semisimple, and we have an orthogonal decomposition

$$\mathsf{Rep}(G) = \bigoplus_{\chi\in\Lambda^T_+} \mathsf{Vect}\otimes V_\chi \ ,$$

where V_χ is the irreducible G-representation with highest weight χ, Vect is the category of finite-dimensional $\mathbb{C}$-vector spaces. Again, we may see the set Λ^T_+ as the quotient Λ^T/W, and after rationalization, we may identify it with Λ_T/W by choosing a non-degenerate W-invariant bilinear form, which always exists.

2.2. Stacks and the component lattice

It is now well-understood that some moduli spaces in algebraic geometry are more naturally seen as *algebraic stacks* as opposed to schemes, especially when the objects that the moduli spaces parametrize have non-trivial automorphisms.

An *algebraic stack* can be roughly defined as a *groupoid*, meaning a category whose morphisms are all invertible, whose sets of objects and morphisms both have the extra structure of algebraic schemes. See, for example, [59] for a modern treatment of algebraic stacks. For example, for a complex algebraic group G, the *classifying stack* $*/G$ has a single object whose automorphism group is G.

Now let $\mathcal{X}$ be a complex algebraic stack. In reality, we usually take $\mathcal{X}$ to be the moduli stack of a certain type of geometric objects, such as of sheaves or of principal bundles over a given complex variety.

Definition 2.2 ([20]). The *component lattice* of a complex algebraic stack $\mathcal{X}$, as a set, is defined by

$$\mathrm{CL}(\mathcal{X}) = \pi_0\big(\mathcal{M}ap(*/\mathbb{C}^\times,\mathcal{X})\big)\,.$$

Here, we take the set of connected components of the mapping stack from $*/\mathbb{C}^\times$ to $\mathcal{X}$.

In other words, since a map from $*/\mathbb{C}^\times$ to $\mathcal{X}$ is the same data as a point $x \in \mathcal{X}$ together with a group homomorphism $\lambda \colon \mathbb{C}^\times \to \mathrm{Aut}(x)$ up to conjugation, we can think of $\mathrm{CL}(\mathcal{X})$ as the set of such pairs (x, λ) up to continuous deformation.

In fact, $\mathrm{CL}(\mathcal{X})$ has a similar structure to Λ_T/W discussed previously, except that $\mathrm{CL}(\mathcal{X})$ can also be glued from multiple copies of lattices $\mathbb{Z}^n$ along sublattices. For example, we have

$$\mathrm{CL}(\mathbb{CP}^1/\mathbb{C}^\times) \simeq \mathbb{Z} \underset{\{0\}}{\cup} \mathbb{Z} \, ,$$

which looks like a cross, where $\mathbb{C}^\times$ acts on $\mathbb{CP}^1$ by scaling, with two fixed points 0 and ∞, which correspond to the two copies of $\mathbb{Z}$.

It turns out that $\mathrm{CL}(\mathcal{X})$ always has a lattice-like combinatorial structure, which we describe in the next section.

2.3. Formal lattices

The combinatorial structure of the component lattice is that of a *formal lattice.*

Definition 2.3 ([20]). A *formal lattice* is a functor

$$L \colon (\mathbb{Z}\text{-}\mathsf{Mod}^{\mathrm{ff}})^{\mathrm{op}} \longrightarrow \mathsf{Set} \, ,$$

where $\mathbb{Z}\text{-}\mathsf{Mod}^{\mathrm{ff}}$ is the category of finitely generated free $\mathbb{Z}$-modules.

For example, via the Yoneda embedding, every object $\mathbb{Z}^n \in \mathbb{Z}\text{-}\mathsf{Mod}^{\mathrm{ff}}$ can be seen as a formal lattice. Also, since the category of formal lattices has arbitrary colimits and limits, we can start from copies of $\mathbb{Z}^n$ and glue them along sublattices, or take quotients by group actions, etc. Just like we can start from affine schemes and glue them into schemes, we can start from lattices $\mathbb{Z}^n$ and glue them into formal lattices.

Given a formal lattice L, we define its *underlying set* to be $|L| = L(\mathbb{Z})$. In this sense, the component lattice $\mathrm{CL}(\mathcal{X})$ can be extended to a formal lattice as follows:

Definition 2.4 ([20]). Let $\mathcal{X}$ be a complex algebraic stack. The component lattice of $\mathcal{X}$, now as a formal lattice, is defined by

$$\mathrm{CL}(\mathcal{X})(\Lambda) = \pi_0\big(\mathcal{M}\mathit{ap}(*/T_\Lambda, \mathcal{X})\big) \, ,$$

where Λ is a finitely generated free $\mathbb{Z}$-module, and T_Λ is the torus whose cocharacter lattice is Λ.

Under mild conditions, including that $\mathcal{X}$ is of finite type, the formal lattice $\mathrm{CL}(\mathcal{X})$ can be glued from finitely many lattices $\mathbb{Z}^n$ along finitely may maps, as in [20, Theorem 6.2.3].

It turns out to be useful to look at maps of formal lattices of the form $\alpha \colon \mathbb{Z}^n \to \mathrm{CL}(\mathcal{X})$, which are in bijection with elements $\alpha \in \mathrm{CL}(\mathcal{X})(\mathbb{Z}^n)$. Such maps are called *faces* of $\mathcal{X}$, and they form a category $\mathsf{Face}(\mathcal{X})$ whose morphisms are maps $\mathbb{Z}^m \to \mathbb{Z}^n$ compatible with the maps to $\mathrm{CL}(\mathcal{X})$.

For example, when $\mathcal{X} = {*}/G$, by looking at the face $\alpha \colon \Lambda_T \to \Lambda_T / W$ given by the quotient map, we can recover the Weyl group as the automorphism group

$$W = \mathrm{Aut}(\alpha) .$$

Note that although Λ_T / W is not defined as a stack – the functor takes values in sets instead of groupoids – it still captures the information of the group that was divided by. Similarly, automorphism groups of faces of general stacks behave like Weyl groups, and are a natural generalization of Weyl groups to stacks.

2.4. Walls and chambers

Apart from the structure of a formal lattice, the component lattice $\mathrm{CL}(\mathcal{X})$ carries another interesting structure, that is a partition into walls and chambers.

For example, when $\mathcal{X} = {*}/G$ for a reductive group G, so that $\mathrm{CL}(\mathcal{X}) = \Lambda_T / W$, roots of G define dual hyperplanes in Λ_T, giving the *root hyperplane arrangement.* For each cocharacter $\lambda \in \Lambda_T$, we may consider the subgroups

$$\begin{aligned} G^{\lambda} &= \{ g \in G \mid \lambda(t) \, g \, \lambda(t)^{-1} = g \text{ for all } t \}, \\ G^{\lambda,+} &= \{ g \in G \mid \lim_{t \to 0} \lambda(t) \, g \, \lambda(t)^{-1} \text{ exists for all } t \}, \end{aligned}$$

called the *Levi subgroup* and the *parabolic subgroup* associated to λ. When $G = \mathrm{GL}_n$, for example, if $\lambda(t) = \mathrm{diag}(t^{k_1}, \ldots, t^{k_n})$ with $k_i \in \mathbb{Z}$ and $k_1 \geq \cdots \geq k_n$, then G^{λ} is the subgroup of block-diagonal matrices, and $G^{\lambda,+}$ is the subgroup of block upper-triangular matrices, where each block corresponds to a group of k_i that are equal.

An elementary observation here is that the subgroups G^{λ} and $G^{\lambda,+}$ only depend on the region that λ belongs to, where we partition $\Lambda_T \otimes \mathbb{R}$ into regions, and each region is a maximal connected subset of $\Lambda_T \otimes \mathbb{R}$ such that any root hyperplane is either disjoint from or contains it.[1]

For a general complex algebraic stack $\mathcal{X}$, we consider similar stacks

$$\begin{aligned} \mathcal{X}_{\lambda} &\subset \mathcal{M}\mathit{ap}({*}/\mathbb{C}^{\times}, \mathcal{X}), \\ \mathcal{X}_{\lambda}^{+} &\subset \mathcal{M}\mathit{ap}(\mathbb{C}/\mathbb{C}^{\times}, \mathcal{X}), \end{aligned}$$

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[1] In fact, one can allow larger, disconnected regions by dropping the connectedness condition, and the statement is still true for G^{λ}, but not for $G^{\lambda,+}$.

which are connected components corresponding to λ, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ by scaling. These mapping stacks were introduced by Halpern-Leistner [35], and called the stacks of *graded points* and *filtrations* of $\mathcal{X}$, respectively. For example, when $\mathcal{X} = */G$, we have $(*/G)_\lambda \simeq */G^\lambda$ and $(*/G)^+_\lambda \simeq */G^{\lambda,+}$.

The stacks $\mathcal{X}_\lambda$ and $\mathcal{X}^+_\lambda$ are crucial to enumerative geometry. There is a correspondence

$$\mathcal{X}_\lambda \xleftarrow{q} \mathcal{X}^+_\lambda \xrightarrow{p} \mathcal{X} \, ,$$

with the arrows induced by the morphisms $0 \colon */\mathbb{C}^\times \to \mathbb{C}/\mathbb{C}^\times$ and $1 \colon * \to \mathbb{C}/\mathbb{C}^\times$. The operation $\star_\lambda = p_* \, q^*$ is called *Hall induction*, and can be defined on cohomology, K-theory, categories of coherent sheaves, etc., under certain conditions. All types of *Hall algebras* are constructed this way.

A general result [20, Theorem 6.1.2] extends the above observation for Lie groups to general stacks. It states that, under very mild conditions, one can partition the component lattice $\mathrm{CL}(\mathcal{X})$ into regions, such that the above correspondence only depends on the region. Any given face of $\mathrm{CL}(\mathcal{X})$ only intersects with finitely many such regions.

This result is important, because it gives a complete characterization of all Hall inductions that can be constructed from a given stack, and it encodes them in a combinatorial structure.

3. Applications

3.1. Donaldson–Thomas invariants

As we mentioned in §1.2 (3), *Donaldson–Thomas invariants*, in their original narrow sense of [28, 77], are numbers associated to moduli stacks of coherent sheaves on a given Calabi–Yau threefold. These numbers generalize the virtual fundamental class of the moduli stack, whose virtual dimension is always zero, and the virtual class makes sense when the moduli stack is not genuinely stacky.

It was shown by Behrend [10] that in this non-stacky case, this number can be computed as a weighted Euler characteristic of the moduli space. This reduces the problem of defining Donaldson–Thomas invariants to the problem of making sense of the Euler characteristic of a stack.

However, as mentioned in §1.1, it is not easy to make a good definition that always gives a finite number. For linear moduli stacks, this was done in a series of technical works [42, 43, 44, 45, 46, 48, 52].

The main idea of these works is to work in a *ring of motives*, that is a ring generated by symbols $[X]$ for $\mathbb{C}$-varieties X, with the relation $[X] = [Z] + [X \setminus Z]$ for closed subvarieties

$Z \subset X$. The class of $\mathbb{A}^1$ is denoted by $\mathbb{L} = [\mathbb{A}^1]$, so that we have $[\mathbb{P}^1] = \mathbb{L} + 1$, etc. Taking the Euler characteristic specializes to $\mathbb{L} = 1$.

We can also associate motives to stacks. For example, $[*/\mathbb{C}^\times] = 1/(\mathbb{L} - 1)$. The problem is that it has a pole at $\mathbb{L} = 1$, and gives infinity when taking the Euler characteristic.

The technical works mentioned above provide a way to add correction terms to the motive $[\mathcal{X}]$ when $\mathcal{X}$ is a linear moduli stack, so that they cancel out the poles at $\mathbb{L} = 1$, and give a motive $\epsilon(\mathcal{X})$ with no poles at $\mathbb{L} = 1$, so the Euler characteristic is well-defined. This result is also called the *no-pole theorem.*

In [21], with the help of intrinsic enumerative geometry, we are able to generalize the no-pole theorem to a very general class of stacks. The idea is to define the motive $\epsilon(\mathcal{X})$ by the requirement that

$$[\mathcal{X}] = \int_{\lambda \in \mathrm{CL}(\mathcal{X})} \star_\lambda(\epsilon(\mathcal{X}_\lambda))\, d\mu(\lambda)\,,$$

where μ is a *stability measure* on $\mathcal{X}$, or roughly a measure on the component lattice, assigning numbers to cones in the component lattice, and $\star_\lambda$ is the Hall induction map (see §2.4) on rings of motives. The generalized no-pole theorem states that $\epsilon(\mathcal{X})$ has no poles at $\mathbb{L} = 1$.

In fact, because of the structure described in §2.4, this integral can often be written as a finite sum, with each term corresponding to a region in the partition of $\mathrm{CL}(\mathcal{X})$ mentioned there, weighted by the measure of the region.

As a consequence, in [21], we are able to define Donaldson–Thomas invariants for a large class of (-1)*-shifted symplectic stacks* in the sense of [70]. This extra structure is required for Behrend's [10] construction and its generalization to stacks in [11, §5] and [16].

These invariants can be interesting outside of type A. For example, in [17], we discuss invariants counting orthogonal or symplectic sheaves on Calabi–Yau threefolds, which should be related to orientifold string theory. We also have a version of Vafa–Witten invariants in type B/C/D, and as mentioned in §1.3, it will be interesting to explore their properties such as modularity and Langlands duality.

3.2. Cohomology of stacks

Motivated by cohomological Donaldson–Thomas theory, as in §1.2 (4), we study decompositions of the cohomology of stacks, with the goal of describing the infinite-dimensional cohomology using a finite amount of data. This is achieved in the following theorem, which is a generalization of §2.1 (2).

Theorem 3.1 ([19, 36]). *Let $\mathcal{X}$ be a complex algebraic stack of finite type that is smooth, almost symmetric, and satisfies other mild conditions. Then we have a decomposition*

$$\mathrm{H}^\bullet(\mathcal{X};\mathbb{Q}) \simeq \bigoplus_{\alpha\colon \text{special face}} \Bigl(\mathrm{H}^\bullet_{\mathrm{BPS}}(\mathcal{X}_\alpha) \otimes \mathbb{Q}[t_1, \dots, t_{\dim\alpha}]\Bigr)^{\mathrm{Aut}(\alpha)} ,$$

which is a finite direct sum, and each $\mathrm{H}^\bullet_{\mathrm{BPS}}(\mathcal{X}_\alpha) \subset \mathrm{H}^\bullet(\mathcal{X}_\alpha;\mathbb{Q})$ is a finite-dimensional subspace, called the BPS cohomology.

Here, being *almost symmetric* means that at any closed point $x \in \mathcal{X}$, the tangent space $\mathrm{T}_x\mathcal{X} = \mathrm{H}^0(\mathbb{T}_{\mathcal{X}}|_x)$ has symmetric weights as a representation of the unit component of the automorphism group $\mathrm{Aut}(x)$.

A *special face* means roughly a face of the component lattice (see §2.3) that is a union of regions described at the end of §2.4, and $\mathcal{X}_\alpha$ means $\mathcal{X}_\lambda$ for a generic λ belonging to the face. The inclusion map $\mathrm{H}^\bullet_{\mathrm{BPS}}(\mathcal{X}_\alpha;\mathbb{Q}) \to \mathrm{H}^\bullet(\mathcal{X};\mathbb{Q})$ is given by Hall induction (see §2.4).

Moreover, there is also a version of Theorem 3.1 for (-1)-*shifted symplectic stacks* $\mathcal{X}$ in the sense of [70], instead of smooth stacks, where we take the *critical cohomology*

$$\mathrm{H}^\bullet_{\mathrm{crit}}(\mathcal{X}) = \mathrm{H}^\bullet(\mathcal{X}, \varphi_{\mathcal{X}}) ,$$

where $\varphi_{\mathcal{X}}$ is a perverse sheaf on $\mathcal{X}$ constructed in [11], sometimes called the *Donaldson–Thomas sheaf.* It is locally given by the vanishing cycle sheaf of a function on a smooth stack, and $\mathcal{X}$ is locally modelled by the critical locus of the same function.

Theorem 3.2 ([19, 36]). *Let $\mathcal{X}$ be an oriented (-1)-shifted symplectic derived algebraic stack of finite presentation over $\mathbb{C}$ that is almost symmetric and satisfies other mild conditions. Then we have a decomposition*

$$\mathrm{H}^\bullet_{\mathrm{crit}}(\mathcal{X}) \simeq \bigoplus_{\alpha\colon \text{special face}} \Bigl(\mathrm{H}^\bullet_{\mathrm{BPS}}(\mathcal{X}_\alpha) \otimes \mathbb{Q}[t_1, \dots, t_{\dim\alpha}]\Bigr)^{\mathrm{Aut}(\alpha)} ,$$

which is a finite direct sum, and each $\mathrm{H}^\bullet_{\mathrm{BPS}}(\mathcal{X}_\alpha) \subset \mathrm{H}^\bullet_{\mathrm{crit}}(\mathcal{X}_\alpha;\mathbb{Q})$ is a finite-dimensional subspace, called the BPS cohomology.

Again, the inclusion map $\mathrm{H}^\bullet_{\mathrm{BPS}}(\mathcal{X}_\alpha;\mathbb{Q}) \to \mathrm{H}^\bullet_{\mathrm{crit}}(\mathcal{X};\mathbb{Q})$ is given by Hall induction, which is well-defined by the work of Kinjo–Park–Safronov [50].

The vector space $\mathrm{H}^\bullet_{\mathrm{BPS}}(\mathcal{X})$ can be seen as a categorification of the Donaldson–Thomas invariant of $\mathcal{X}$, and its dimension is related to the latter by a multiple cover formula [22].

3.3. Coherent sheaves on stacks

The decomposition theorems in cohomology in §3.2 can be categorified even further, giving decompositions of derived categories of coherent sheaves on stacks. This generalizes the simple observation in §2.1 (3), and extends *categorical Donaldson–Thomas theory*, as in §1.2 (5), to more general stacks.

Theorem 3.3 ([24]). *Let $\mathcal{X}$ be a derived algebraic stack of finite presentation over $\mathbb{C}$ that is quasi-smooth, quasi-symmetric, and satisfies other mild conditions. Then there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{X}) = \big\langle \mathsf{W}_{\mathcal{X}_\lambda}(\delta_\lambda) \bigm| \lambda \in \mathrm{CL}(\mathcal{X}) \otimes \mathbb{Q} \big\rangle \, ,$$

where each $\mathsf{W}_{\mathcal{X}_\lambda}(\delta_\lambda) \subset \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{X}_\lambda)$ is a full subcategory, called a window subcategory, and the inclusion functor $\mathsf{W}_{\mathcal{X}_\lambda}(\delta_\lambda) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{X})$ is given by Hall induction.

Here, being *quasi-symmetric* is slightly weaker than being almost symmetric. Having a *semiorthogonal decomposition* means that every object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{X})$ has a unique filtration with stepwise quotients lying in the semiorthogonal summands, in a fixed order. In Theorem 3.3, this order is given by the reverse order of $|\lambda|_q \in \mathbb{R}_{\geq 0}$, the norm of λ with respect to a positive-definite quadratic form q on $\mathrm{CL}(\mathcal{X}) \otimes \mathbb{Q}$.

Assuming that $\mathcal{X}$ is connected, the leading term $\mathsf{W}_{\mathcal{X}}(\delta)$ in the decomposition can be seen as a categorical Donaldson–Thomas invariant of $\mathcal{X}$. It is expected that one recovers information about BPS cohomology by taking cohomological or K-theoretic invariants, such as periodic cyclic homology or Blanc's topological K-theory [12]; the latter has been studied in [65, 68] in the linear case.

There is also a categorical analogue of the critical cohomology mentioned in §3.2. Given a (-1)-shifted symplectic stack $\mathcal{X}$, if $\mathcal{X}$ is the derived critical locus of a function $f \colon \mathcal{U} \to \mathbb{C}$ on a smooth stack $\mathcal{U}$, then this analogue is the category $\mathsf{MF}(\mathcal{U}, f)$ of *matrix factorizations* of f. It is not known how to define such a category for a general (-1)-shifted symplectic stack $\mathcal{X}$, although substantial progress has been made by Hennion–Holstein–Robalo [37, 38]. But in the case of a derived critical locus, we have the following decomposition.

Theorem 3.4 ([24]). *Let $\mathcal{U}$ be a complex algebraic stack of finite type that is smooth, quasi-symmetric, and satisfies other mild conditions, and let $f \colon \mathcal{U} \to \mathbb{C}$ be a function. Then there is a semiorthogonal decomposition*

$$\mathsf{MF}(\mathcal{U}, f) = \big\langle \mathsf{M}_{\mathcal{U}_\lambda, f}(\delta_\lambda) \bigm| \lambda \in \mathrm{CL}(\mathcal{U}) \otimes \mathbb{Q} \big\rangle \, ,$$

where each $\mathsf{M}_{\mathcal{X}_\lambda, f}(\delta_\lambda) \subset \mathsf{MF}(\mathcal{U}_\lambda, f)$ is a full subcategory called a window subcategory, and the inclusion functor is given by Hall induction.

The decomposition in Theorem 3.3 also has direct applications to the geometric Langlands programme. Namely, after taking the ind-completion, we obtain decompositions of the categories

$$\mathsf{Ind}\,\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{H}\!iggs_G^{\mathrm{ss}})\ , \qquad \mathsf{Ind}\,\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{C}\!onn_G)\ , \qquad \mathsf{Ind}\,\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{L}\!oc_G)$$

of *ind-coherent sheaves* as in [29, 32], where we fix a smooth projective curve C and a reductive group G, and

- $\mathscr{H}\!iggs_G^{\mathrm{ss}}$ is the moduli stack of semistable G-Higgs bundles on C;
- $\mathscr{C}\!onn_G$ is the moduli stack of principal G-bundles with holomorphic connections on C;
- $\mathscr{L}\!oc_G$ is the moduli stack of G-local systems on C.

In [24], we also prove decompositions of subcategories of objects with *nilpotent singular support* as in [9]. These categories appear on the spectral side of the three versions of the geometric Langlands conjecture: The Dolbeault version, formulated in [69] and partially proved for GL_2 by Toda [80]; and the de Rham and Betti versions, proved in [7, 8, 25, 33, 34].

Thus, in all three versions, we decompose the category on one side of the geometric Langlands correspondence, and it is interesting to study the corresponding decomposition on the other side, and to describe it geometrically. These enable us to formulate refined versions of the geometric Langlands conjectures which help us further understand the structure of the correspondence; see [69] for more discussions on the Dolbeault case and [24] for the de Rham case.

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`bucj@mailbox.org`
Mathematical Institute, University of Oxford, Oxford OX2 6GG, United Kingdom.