Stable Irrationality of Varieties

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ABSTRACT

We present several methods of showing the existence of stably irrational varieties within a given family, and we show the infinitude of stable birational classes in the family of quartic threefolds.

Contents

1 Introduction

For a projective variety, there are various notions of rationality, describing how close a variety is to the projective space.

Definition 1.1. Let $\mathbf{k}$ be a field, and let X be a projective $\mathbf{k}$-variety.

- X is *rational*, if there exists $n \in \mathbf{N}$, such that
$$X \quad \text{is birational to} \quad \mathbf{P}^n_{\mathbf{k}}.$$
Equivalently, we have $\mathbf{k}(X) \simeq \mathbf{k}(x_1, \dots, x_n)$ as $\mathbf{k}$-algebras.

- X is *stably rational*, if there exist $m, n \in \mathbf{N}$, such that
$$X \times \mathbf{P}^m_{\mathbf{k}} \quad \text{is birational to} \quad \mathbf{P}^n_{\mathbf{k}}.$$
Equivalently, we have $\mathbf{k}(X)(y_1, \dots, y_m) \simeq \mathbf{k}(x_1, \dots, x_n)$ as $\mathbf{k}$-algebras.

- X is *retract rational*, if there exists $n \in \mathbf{N}$, and open sets $U \subset X$, $V \subset \mathbf{P}^n_{\mathbf{k}}$, together with two maps
$$f : U \to V, \quad g : V \to U,$$
such that $g \circ f = \mathrm{id}_U$.

- X is *unirational*, if there exists a dominant rational map
$$\mathbf{P}^n_{\mathbf{k}} \dashrightarrow X.$$
Equivalently, there exists a map $\mathbf{k}(X) \to \mathbf{k}(x_1, \dots, x_n)$ of $\mathbf{k}$-algebras.

- X is *rationally connected*, if for every algebraically closed field $\mathbf{K}$ containing $\mathbf{k}$, and any two $\mathbf{K}$-points $x, y \in X(\mathbf{K})$, there exists a rational curve
$$f : \mathbf{P}^1_{\mathbf{K}} \to X_{\mathbf{K}}$$
joining them, i.e. we have $f(0) = x$ and $f(\infty) = y$.

These notions are sorted from strong to weak, i.e. every notion implies its next one. A natural question to ask is that whether these implications are strict.

In 1972, Artin and Mumford [AM72] gave an example of a variety that is unirational but not retract rational. More recently, Voisin [Voi15] developed a deformation method which can show that a very general member of a family of varieties is not retract rational, as long as it contains one particular example that is not retract rational. This method was then modified by Colliot-Thélène and Pirutka [CTP16] to show that over $\mathbf{C}$, a very general quartic hypersurface in $\mathbf{P}^4$ is not retract rational. They also developed the specialisation method, with which they can provide more general examples of smooth varieties that are not retract rational over number fields and local fields.

In this article, we give an exposition of the methods and results mentioned above, and we show the infinitude of stable birational classes in the family of quartic threefolds (Theorem 5.7).

2 Criteria for rationality

In this section, we relate rationality with several other invariants of a variety. We show that retract rationality implies that these invariants are trivial. The results are summarised in the following diagram, although some assumptions are dropped:

$$\begin{array}{ccccccc} \text{retract} & & \text{universally} & & \text{decomposition} & & \text{trivial} \\ \text{rational} & \Longrightarrow & \mathrm{CH}_0\text{-trivial} & \Longleftrightarrow & \text{of the diagonal} & \Longrightarrow & \text{Brauer group.} \end{array}$$

Chow groups

In this subsection, we recall the definition and some basic properties of the Chow groups of a variety. A general reference is [Ful98].

In the following, let $\mathbf{k}$ be a field, and let X be a $\mathbf{k}$-variety.

Definition 2.1. Let d be an non-negative integer. The free abelian group

$$\mathrm{Z}_d(X) = \bigoplus_{Z\subset X} \mathbf{Z}\cdot[Z],$$

where Z runs through all d-dimensional integral closed subvarieties of X, is called the group of *d-cycles* of X.

In other words, a d-cycle of X is a finite sum $\sum n_i[Z_i]$, where each n_i is an integer, and each Z_i is a subvariety of X. For example, a 0-cycle is a linear combination of closed points.

Let $V \subset X$ be a $(d+1)$-dimensional integral closed subvariety, and let $f \in \mathbf{k}(V)^\times$ be a rational function on V. The principal divisor of V corresponding to f, denoted by $\mathrm{div}(f)$, can be naturally seen as a d-cycle of X. This defines a map of abelian groups

$$\mathrm{div}:\ \mathbf{k}(V)^\times \to \mathrm{Z}_d(X).$$

Definition 2.2. The d-th *Chow group* of X is defined by

$$\mathrm{CH}_d(X) = \mathrm{coker}\Big(\bigoplus_{V\subset X} \mathbf{k}(V)^\times \to \mathrm{Z}_d(X)\Big),$$

where V runs through all $(d+1)$-dimensional integral closed subvarieties of X.

If X has dimension n, we write

$$\mathrm{Z}^d(X) = \mathrm{Z}_{n-d}(X) \quad \text{and} \quad \mathrm{CH}^d(X) = \mathrm{CH}_{n-d}(X).$$

An element of the Chow group is thus a class of cycles. We say that the cycles in the same class are *rationally equivalent*.

Here we explain four operations of the Chow group: proper pushforward, flat pullback, the intersection product, and the Gysin map.

Definition 2.3. Let $f : X \to Y$ be a proper map of $\mathbf{k}$-varieties, and let $Z \subset X$ be an integral closed subvariety of dimension d. We define

$$f_*[Z] = \begin{cases} 0, & \text{if } \dim f(Z) < d, \\ [\mathbf{k}(f(Z)) : \mathbf{k}(Z)]\,[f(Z)], & \text{if } \dim f(Z) = d, \end{cases}$$

as a d-cycle of Y, where

- $f(Z)$ is an irreducible closed subset of Y, which we see as an integral closed subscheme.
- The field extension $\mathbf{k}(f(Z))/\mathbf{k}(Z)$ is finite because it is finitely generated and of transcendence degree 0.

This extends linearly to a *pushforward map*

$$f_* : \mathrm{Z}_d(X) \to \mathrm{Z}_d(Y).$$

It turns out that proper pushforward preserves rational equivalence [Ful98, §1.4]. We thus obtain a *pushforward map* of Chow groups

$$f_* : \mathrm{CH}_d(X) \to \mathrm{CH}_d(Y).$$

Next, we introduce the flat pullback of Chow groups.

Definition 2.4. Let $f : X \to Y$ be a map of $\mathbf{k}$-varieties, which is flat of relative dimension r. Let $Z \subset X$ be an integral closed subvariety of dimension d. We define

$$f^*[Z] = [f^{-1}(Z)]$$

as a $(d + r)$-cycle of Y, where

- $f^{-1}(Z)$ is the scheme-theoretic inverse image, i.e., $Z \times_Y X$.
- $[f^{-1}(Z)]$ denotes the sum $\sum m_i[Z_i]$, where the Z_i are the irreducible components of $f^{-1}(Z)$, and m_i is the *geometric multiplicity* of Z_i in $f^{-1}(Z)$, defined as the length of the local ring $\mathcal{O}_{f^{-1}(Z),Z_i}$.

This extends linearly to a *pullback map*

$$f^* : \mathrm{Z}_d(Y) \to \mathrm{Z}_{d+r}(X).$$

It turns out again that flat pullback preserves rational equivalence [Ful98, §1.7]. Switching to the cohomological indexing notation, we get a *pullback map* of Chow groups

$$f^* : \mathrm{CH}^d(Y) \to \mathrm{CH}^d(X).$$

Now, we introduce the intersection product of cycles.

Let $Z_1, Z_2 \subset X$ be two integral closed subvarieties. We can form their "scheme-theoretic intersection"

$$Z_1 \cap Z_2 = Z_1 \times_X Z_2.$$

However, this does not always produce cycles of the expected dimension, as the subvarieties may not be in a general position to intersect. To work around this difficulty, we use the Gysin map of the diagonal map, which acts as a pullback along a closed embedding.

The Gysin map is defined for vector bundles as follows.

Theorem 2.5. *Let $p : E \to X$ be a vector bundle of rank r. Then the flat pullback*

$$p^* : \mathrm{CH}_d(X) \to \mathrm{CH}_{d+r}(E)$$

is an isomorphism for all d. Its inverse is called the Gysin map*, and denoted by $i^!$, where i is the zero section map $X \to E$.*

See [Ful98, §3.3].

Recall that a *regular embedding* is a closed embedding of schemes, such that the ideal sheaf is locally generated by regular sequences. For example, a closed embedding of smooth varieties is always a regular embedding.

For a regular embedding, the normal cone is a vector bundle. We can use this property to extend the definition of the Gysin map to this case.

Definition 2.6. Let $i : Z \to X$ be a regular embedding of constant codimension e. The *Gysin map*

$$i^! : \mathrm{CH}_d(X) \to \mathrm{CH}_{d-e}(Z)$$

is defined as follows. Let $N_Z X$ denote the normal bundle of Z in X, and let

$$\sigma : \mathrm{Z}_d(X) \to \mathrm{Z}_d(N_Z X)$$

be the map given by

$$[V] \mapsto [N_{Z\cap V} V].$$

This map respects rational equivalence [Ful98, §5.2], inducing a map

$$\sigma : \mathrm{CH}_d(X) \to \mathrm{CH}_d(N_Z X)$$

We then compose this map with the Gysin map defined in Theorem 2.5, giving the desired map

$$i^! : \mathrm{CH}_d(X) \to \mathrm{CH}_{d-e}(Z).$$

Using the Gysin map as a pullback along the diagonal map, we can define the intersection product of cycles.

Definition 2.7. Let X be an n-dimensional projective variety. The *intersection product* is a map of graded abelian groups

$$\cdot : \mathrm{CH}^\bullet(X) \otimes \mathrm{CH}^\bullet(X) \to \mathrm{CH}^\bullet(X),$$

defined as follows.

- If X is smooth, we define this map by the composition

$$\mathrm{CH}_{d_1}(X) \otimes \mathrm{CH}_{d_2}(X) \xrightarrow{\times} \mathrm{CH}_{d_1+d_2}(X \times X) \xrightarrow{\Delta^!} \mathrm{CH}_{d_1+d_2-n}(X),$$

  where $\times$ denotes the cross product map sending $[Z_1] \otimes [Z_2]$ to $[Z_1 \times Z_2]$, and $\Delta : X \to X \times X$ is the diagonal map, which is a regular embedding.

- If X is arbitrary, we can always embed X in some $\mathbf{P}^N$, so that we can regard cycles of X as cycles of $\mathbf{P}^N$, and intersect them in $\mathbf{P}^N$.

This product equips the Chow groups with the structure of a graded ring, called the *Chow ring*.

Rationality and zero-cycles

Definition 2.8. We say that a map $f : X \to Y$ of $\mathbf{k}$-varieties is *universally* CH_0-*trivial*, if

- f is proper.
- For any field extension $F/\mathbf{k}$, the pushforward map

$$f_* : \mathrm{CH}_0(X_F) \to \mathrm{CH}_0(Y_F)$$

  is an isomorphism.

If $Y = \operatorname{Spec} \mathbf{k}$, then we say X is *universally* CH_0-*trivial*. This means that

- X is complete.
- For any field extension $F/\mathbf{k}$, the degree map

$$\deg_F : \mathrm{CH}_0(X_F) \to \mathbf{Z}$$

  is an isomorphism.

We will show that retract rationality implies universal CH_0 triviality. The proof will need a few lemmas. First of all, we prove a moving lemma for 0-cycles.

Lemma 2.9. *Let X be a smooth projective* $\mathbf{k}$*-variety, with* $\mathbf{k}$ *infinite and perfect, and let $U \subset X$ be a dense open set. Then every* 0*-cycle of X is rationally equivalent to one supported in U.*

Proof. We follow [CT05, Complément]. Write $Z = X \setminus U$, and let $p \in Z$ be a closed point. It suffices to show that the 0-cycle $[p]$ is equivalent to one supported in U.

Let $g \in \mathcal{O}_{X,p}$ be a locally defined non-zero function that vanishes on Z. Since X is smooth, we can find a regular sequence $f_1, \dots, f_{n-1}$ of $\mathcal{O}_{X,p}$, where $n = \dim X$, such that $g \neq 0$ in the quotient $\mathcal{O}_{X,p}/(f_1, \dots, f_{n-1})$. This can be done by working in affine coordinates and taking the f_i to be linear functions.

The ideal $(f_1, \dots, f_{n-1})$ defines a curve in a neighbourhood of p. Taking its closure in X, we obtain a closed integral curve C in X, which is regular at p and is not contained in Z.

Let $f : D \to C$ be the normalisation of C. Then D is regular (normality implies regularity in codimension one), and hence smooth since $\mathbf{k}$ is perfect. Also, D is quasi-projective [EGA-II, Corollary 7.4.10], and f is finite [EGA-II, Corollary 7.4.6], and hence proper [EGA-II, Corollary 6.1.11].

Let q be the inverse image of p, which is a single point as C is regular at p. Let W be a neighbourhood of q in D, such that $f|_W$ is an isomorphism. For example, we can take W to be the inverse image of the smooth locus of C.

The 0-cycle $[q]$ is equivalent to a 0-cycle supported in $W \setminus f^{-1}(Z)$. Indeed, we have to find a rational function on D that has a simple zero at q, and is defined on the finite set $(D \setminus W) \cup f^{-1}(Z)$. As D is quasi-projective, this can be done by taking a suitable linear function on the projective space.

Finally, we consider the pushforward along the proper map $D \to C \to X$. Since it preserves rational equivalence, we are done. □

Remark 2.10. This is a special case of [EGA-II, Proposition 7.4.9], but the proof given here is more elementary.

Lemma 2.11. *Let* $\mathbf{k}$ *be a finite field, and let $U \subset \mathbf{P}^n_{\mathbf{k}}$ be a non-empty open set. Then for any extension $F/\mathbf{k}$ of sufficiently large degree d, the open set $U_F \subset \mathbf{P}^n_F$ contains an F-rational point.*

Proof. Let q be the cardinality of $\mathbf{k}$. Let f be a non-zero homogeneous polynomial over $\mathbf{k}$, which vanishes outside U.

In $\mathbf{P}^n_F$, when $q^d > \deg f$, there are at least $(q^d - \deg f)^n$ rational points where f does not vanish. Indeed, by induction on n, one easily shows that a non-zero polynomial of degree r on $\mathbf{A}^n_F$ does not vanish at at least $(q^d - r)^n$ rational points, provided that $q^d > r$.

Hence U_F contains a rational point whenever $q^d > \deg f$. □

Theorem 2.12 (Colliot-Thélène and Pirutka)**.** *Let X be a smooth projective* $\mathbf{k}$*-variety. If X is retract rational, then X is universally* CH_0*-trivial.*

Proof. First, we suppose that the base field $\mathbf{k}$ is infinite. Since retract rationality is preserved under change of base field, it suffices to prove that X is CH_0-trivial over $\mathbf{k}$, i.e. $\deg_{\mathbf{k}} : \mathrm{CH}_0(X) \to \mathbf{Z}$ is an isomorphism.

By definition, there exist non-empty open sets $U \subset X$, $V \subset \mathbf{P}^n_{\mathbf{k}}$, and maps

$$U \xrightarrow{f} V \xrightarrow{g} U,$$

whose composition is id_U.

Let $P \in U$ be a closed point, and write $Q = f(P) \in V$. Then we have induced maps of residue fields $\kappa(P) \to \kappa(Q) \to \kappa(P)$, whose composition is $\mathrm{id}_{\kappa(P)}$. Therefore, we have

$$\kappa(P) \simeq \kappa(Q).$$

Let F denote this field. We then consider the diagram

$$\begin{array}{ccc} \mathbf{P}^n_F \supset V_F & \xrightarrow{g_F} & U_F \subset X_F \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathbf{P}^n_{\mathbf{k}} \supset V & \xrightarrow{g} & U \subset X \end{array}.$$

There exists $R \in p^{-1}(Q)$ such that $\kappa(R) \simeq F$. This is because $p^{-1}(Q) \simeq \mathrm{Spec}(F \otimes_{\mathbf{k}} F)$, and we can take R to be the point defined by the maximal ideal which is the kernel of the multiplication map $F \otimes_{\mathbf{k}} F \to F$.

Let $A \in V \subset \mathbf{P}^n_{\mathbf{k}}$ be a $\mathbf{k}$-rational point, which exists since $\mathbf{k}$ is infinite. Let $L \simeq \mathbf{P}^1_F \subset \mathbf{P}^n_F$ be a line connecting R and A_F. The map g_F sends L to a rational line L' in U_F, which is a rational F-map $\mathbf{P}^1_F \dashrightarrow U_F$. This map extends to a map

$$L' : \mathbf{P}^1_F \to X_F,$$

which is proper since $\mathbf{P}^1_F$ is complete [EGA-II, Corollary 5.4.3]. This is a line connecting the points $g_F(R)$ and $g_F(A_F)$.

As R is an F-rational point, we have $\kappa(g_F(R)) \simeq F$ and $\kappa(g_F(A_F)) \simeq F$. Hence, the pushforward map of L' on CH_0 gives

$$[g_F(R)] = [g_F(A_F)] \quad \in \mathrm{CH}_0(X_F).$$

Pushing forward to X, and noticing that $\kappa(g(A)) \simeq \mathbf{k}$, we thus have

$$[P] = [F : \mathbf{k}]\,[g(A)] \quad \in \mathrm{CH}_0(X).$$

By the moving lemma 2.9, every 0-cycle of X is equivalent to one supported in U, which, as we have shown, is equivalent to a multiple of $[g(A)]$. Since $\deg_{\mathbf{k}} [g(A)] = 1$, this shows that X is CH_0-trivial over $\mathbf{k}$.

Finally, if $\mathbf{k}$ is a finite field, by Lemma 2.11, we can apply the above argument to an extension $F/\mathbf{k}$ of sufficiently large degree d. Since the composition

$$\mathrm{CH}_0(X) \xrightarrow{p^*} \mathrm{CH}_0(X_F) \xrightarrow{p_*} \mathrm{CH}_0(X)$$

is multiplication by d, where $p\colon X_F \to X$ is the projection, it follows that every 0-cycle of X of degree 0 is d-torsion in $\mathrm{CH}_0(X)$. Hence it must be zero, since d can be chosen to be two coprime values. Moreover, this also shows that X has two 0-cycles of coprime degrees, and hence, the degree map $\deg\colon \mathrm{CH}_0(X) \to \mathbf{Z}$ is surjective. □

We also mention the following criterion for universal CH_0-triviality of a morphism, which will be useful later.

Theorem 2.13 (Colliot-Thélène and Pirutka)**.** *Let $f : \widetilde{X} \to X$ be a proper morphism of* $\mathbf{k}$*-varieties. Suppose that*

- *For every point $M \in X$, not necessarily closed, the fibre $\widetilde{X}_M$ is universally* CH_0*-trivial as a $\kappa(M)$-variety.*

Then f is universally CH_0*-trivial.*

Proof. It suffices to show that $f_* : \mathrm{CH}_0(\widetilde{X}) \to \mathrm{CH}_0(X)$ is an isomorphism.

By assumption, this map f_* is surjective. Let x be a 0-cycle of $\widetilde{X}$, such that $f_*(x)$ is equivalent to zero. We need to show that x is equivalent to zero.

In this case, there exist finitely many integral curves $C_i \subset X$, and functions $g_i \in \mathbf{k}(C_i)$, such that

$$f_*(x) = \textstyle\sum_i \mathrm{div}_{C_i}(g_i).$$

Let η_i be the generic point of C_i. By hypothesis, each fibre $\widetilde{X}_{\eta_i}$ contains a 0-cycle

$$\textstyle\sum_j n_{ij}[D_{ij}]$$

of degree 1, where $n_{ij} \in \mathbf{Z}$. We regard each D_{ij} as a curve in $\widetilde{X}$. Then each function $g_i \circ f$ defines a rational function g_{ij} on D_{ij}. Write

$$x' = x - \textstyle\sum_{i,j} n_{ij}\, \mathrm{div}_{D_{ij}}(g_{ij}).$$

Then we have an equality of *cycles* $f_*(x') = 0$.

Let us write $x' = \sum_i x'_{Q_i}$, where $Q_i \in X$ are distinct points, and x'_{Q_i} is a 0-cycle of $\widetilde{X}$ supported in the fibre $\widetilde{X}_{Q_i}$. The fact that $f_*(x') = 0$ implies that each x'_{Q_i} has degree 0. It follows from the hypothesis applied to $\widetilde{X}_{Q_i}$ that x'_{Q_i} is rationally equivalent to zero. Therefore, x' is equivalent to zero, and so is x. □

Decomposition of the diagonal

We now give an equivalent characterisation of universal CH_0-triviality. We show that it is equivalent to the existence of a decomposition of the diagonal class in the Chow group of $X \times X$.

Definition 2.14. Let X be a complete $\mathbf{k}$-variety of dimension n. A *decomposition of the diagonal* of X is given by an equation

$$[\Delta_X] = D + [X] \times x_0 \quad \text{in } \mathrm{CH}_n(X \times X),$$

where

- $[\Delta_X]$ is the pushforward of $[X] \in \mathrm{CH}_n(X)$ along the diagonal map $X \to X \times X$.
- D is an n-cycle of $X \times X$, supported in $Z \times X$ for some closed subvariety $Z \subset X$ of codimension at least 1.
- x_0 is a 0-cycle of X of degree 1.

In order to show that this property is equivalent to CH_0-triviality, we introduce the notion of a correspondence.

Definition 2.15. Let X and Y be complete $\mathbf{k}$-varieties, of dimensions m and n, respectively. A *correspondence* from X to Y is an element of the set

$$\mathrm{Corr}(X, Y) = \mathrm{CH}_m(X \times Y).$$

We view a correspondence as a generalised version of a graph of a map from X to Y. In this way, we can compose correspondences as if we are composing graphs of maps. Namely, for $f \in \mathrm{Corr}(X, Y)$ and $g \in \mathrm{Corr}(Y, Z)$, we define

$$g \circ f = p_*\big(([X] \times g) \cdot (f \times [Z])\big) \quad \in \mathrm{Corr}(X, Z),$$

where $p\colon X \times Y \times Z \to X \times Z$ is the projection map.

Moreover, we have a group homomorphism

$$\begin{aligned} \mathrm{Corr}(X, Y) \otimes_{\mathbf{Z}} \mathrm{CH}_\bullet(X) &\to \mathrm{CH}_\bullet(Y), \\ (f, \alpha) &\mapsto p_*\big(f \cdot (\alpha \times [Y])\big), \end{aligned} \tag{2.15.1}$$

where $p\colon X \times Y \to Y$ is the projection map. In particular, this induces an action of $\mathrm{Corr}(X, X)$ on $\mathrm{CH}_\bullet(X)$.

Proposition 2.16. *Complete* $\mathbf{k}$*-varieties and correspondences form a category, which admits a functor from the category of complete* $\mathbf{k}$*-varieties and* $\mathbf{k}$*-maps. The functor* $\mathrm{CH}_\bullet$ *factors through this functor.* □

Before proving the main theorem, we need a lemma.

Lemma 2.17. *Let X be an integral* $\mathbf{k}$*-variety, and let η be its generic point. Consider the map*

$$\eta \times \mathrm{id}_X\colon\ \mathrm{Spec}(\mathbf{k}(X)) \times X \simeq X_{\mathbf{k}(X)} \to X \times X.$$

The pullback of the diagonal class is the class of the generic point of X, which is a 0*-cycle of $X_{\mathbf{k}(X)}$ of degree* 1.

Proof. We may assume $X = \operatorname{Spec} A$ is affine. Then the pullback of the diagonal class is the closed point defined by the maximal ideal, which is the kernel of the multiplication map

$$\mathbf{k}(X) \otimes A \to \mathbf{k}(X).$$

The residue field is thus $\mathbf{k}(X)$. □

Theorem 2.18 (Colliot-Thélène and Pirutka)**.** *Let X be a smooth, integral, complete $\mathbf{k}$-variety. Then the following are equivalent.*

(i) *X is universally CH_0-trivial.*

(ii) *X has a 0-cycle of degree 1, and the degree map $\deg : \mathrm{CH}_0(X_{\mathbf{k}(X)}) \to \mathbf{Z}$ is an isomorphism.*

(iii) *X admits a decomposition of the diagonal.*

Proof. The implication (i) ⇒ (ii) follows from the definition of universal CH_0-triviality.

Assume (ii). Let α be a 0-cycle of X of degree 1, and let $\beta \in \mathrm{CH}_0(X_{\mathbf{k}(X)})$ be the class of the generic point of X. Then by hypothesis, we have

$$\alpha_{\mathbf{k}(X)} = \beta \quad \text{in } \mathrm{CH}_0(X_{\mathbf{k}(X)}).$$

By [Blo10, Lemma 1A.1], we have

$$\mathrm{CH}^n(X_{\mathbf{k}(X)}) \simeq \operatorname*{colim}_{U \subset X \text{ open}} \mathrm{CH}^n(U \times X).$$

By Lemma 2.17, the map $\mathrm{CH}^n(X \times X) \to \mathrm{CH}^n\big(\operatorname{Spec}(\mathbf{k}(X)) \times X\big) \simeq \mathrm{CH}_0(X_{\mathbf{k}(X)})$ maps the diagonal class $[\Delta_X]$ to β. Therefore, there exists a non-empty open set $U \subset X$, such that

$$[U] \times \alpha = [\Delta_U] \quad \text{in } \mathrm{CH}_n(U \times X).$$

Let $Z = X \setminus U$. By [Ful98, §1.8], we have an exact sequence

$$\mathrm{CH}_n(Z \times X) \to \mathrm{CH}_n(X \times X) \to \mathrm{CH}_n(U \times X) \to 0,$$

which implies that there exists $D \in \mathrm{CH}_n(Z \times X)$, such that

$$D = [X] \times \alpha - [\Delta_X] \quad \text{in } \mathrm{CH}_n(X \times X).$$

This proves (iii).

Now assume (iii), and let

$$[\Delta_X] = D + [X] \times x_0 \quad \text{in } \mathrm{CH}_n(X \times X)$$

be a decomposition, with D supported in $Z \times X$. Let F be an extension of $\mathbf{k}$.

Consider the action of $\mathrm{Corr}(X, X)$ on $\mathrm{CH}_0(X_F)$, as defined in (2.15.1). Since $[\Delta_X]$ acts as the identity, so does $D + [X] \times x_0$.

On the other hand, by the moving lemma 2.9, every 0-cycle of X can be moved out of Z. This shows that the action of D is 0, as in the defining equation (2.15.1), we are taking the intersection product of two disjoint cycles. But for any 0-cycle γ of X, the action of $[X] \times x_0$ sends it to

$$p_*\big(([X] \times x_0) \cdot (\gamma \times [X])\big) = p_*(\gamma \times x_0) = (\deg \gamma)\, x_0,$$

where $p\colon X \times X \to X$ is the second projection. This implies that $\mathrm{CH}_0(X_F)$ is generated by x_0, which has degree 1. This proves (i). □

The Brauer group

In this subsection, we show that the existence of a decomposition of the diagonal implies the triviality of the Brauer group.

Definition 2.19. The (cohomological) *Brauer group* of a scheme X is the étale cohomology group

$$\mathrm{Br}(X) = H^2(X, \mathbf{G}_{\mathrm{m}}).$$

For a field $\mathbf{k}$, the Brauer group $\mathrm{Br}(\mathbf{k}) = \mathrm{Br}(\mathrm{Spec}\,\mathbf{k})$ coincides with the classical notion defined as the group of equivalence classes of central simple algebras. A classical reference is [Gro68].

The Kummer exact sequence of étale sheaves

$$0 \to \boldsymbol{\mu}_n \to \mathbf{G}_{\mathrm{m}} \xrightarrow{(-)^n} \mathbf{G}_{\mathrm{m}} \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{Pic}(X) \to H^2(X, \boldsymbol{\mu}_n) \to \mathrm{Br}(X) \xrightarrow{\cdot n} \mathrm{Br}(X) \to \cdots.$$

When $X = \mathrm{Spec}\, R$, where R is a local ring, we have $\mathrm{Pic}(X) = 0$, so that

$$\mathrm{Br}(X)\,[n] \simeq H^2(X, \boldsymbol{\mu}_n),$$

where the left hand side denotes the n-torsion subgroup of $\mathrm{Br}(X)$.

Definition 2.20. Let M be a contravariant functor from schemes to abelian groups (e.g. étale cohomology), and let $\mathbf{k} \subset \mathbf{K}$ be two fields. Let

$$M_{\mathrm{nr}}(\mathbf{K}/\mathbf{k}) = \bigcap_{\mathbf{k} \subset A \subset \mathbf{K}} \mathrm{image}\Big(M(\mathrm{Spec}\, A) \to M(\mathrm{Spec}\, \mathbf{K})\Big),$$

where A runs through all discrete valuation rings with fraction field $\mathbf{K}$.

This is called the *unramified* version of the functor M. For example, one has the *unramified Brauer group* $\mathrm{Br}_{\mathrm{nr}}(\mathbf{K}/\mathbf{k})$, and *unramified cohomology* $H^q_{\mathrm{nr}}(\mathbf{K}/\mathbf{k}, \boldsymbol{\mu}_n)$.

A deep result on the cohomological purity of the Brauer group gives rise to the following theorem.

Theorem 2.21. *Let X be a regular, complete, integral* $\mathbf{k}$*-variety. Then the natural map* $\mathrm{Br}(X) \to \mathrm{Br}(\mathbf{k}(X))$ *induces an isomorphism*

$$\mathrm{Br}(X) \simeq \mathrm{Br}_{\mathrm{nr}}(\mathbf{k}(X)/\mathbf{k}).$$

See [CTS19, Proposition 5.2.2].

The discussion above immediately implies the following.

Corollary 2.22. *Let X be a regular, proper, integral* $\mathbf{k}$*-variety. Then*

$$\mathrm{Br}(X)\,[n] \simeq H^2_{\mathrm{nr}}(\mathbf{k}(X)/\mathbf{k}, \boldsymbol{\mu}_n). \qquad \square$$

Since the Brauer group is a torsion group [CTS19, Proposition 1.3.6], it can thus be computed by the unramified cohomology groups.

On the other hand, the second cohomology of $\boldsymbol{\mu}_n$ is a part of the cycle module (in the sense of [Ros96])

$$\textstyle\bigoplus_i H^i(-, \boldsymbol{\mu}_n^{\otimes(i-1)}),$$

which we do not give a precise definition here. This allows it to be regarded as a "coefficient group" for Chow groups. As a result, there is an action of correspondences

$$\mathrm{Corr}(X, Y) \otimes H^2_{\mathrm{nr}}(\mathbf{k}(X)/\mathbf{k}, \boldsymbol{\mu}_n) \longrightarrow H^2_{\mathrm{nr}}(\mathbf{k}(Y)/\mathbf{k}, \boldsymbol{\mu}_n)$$

for $\mathbf{k}$-varieties X, Y. This action will relate the Brauer group with the decomposition of the diagonal.

Theorem 2.23. *Let X be a smooth projective* $\mathbf{k}$*-variety. If X admits a decomposition of the diagonal, then the natural map induces an isomorphism*

$$\mathrm{Br}(\mathbf{k}) \xrightarrow{\sim} \mathrm{Br}(X).$$

In particular, if $\mathbf{k}$ *is separably closed, then* $\mathrm{Br}(X) = 0$.

Sketch of proof. The decomposition of the diagonal implies that the identity map and the constant map (to be precise, a sum of constant maps) induce the same action on

$$H^2_{\mathrm{nr}}(\mathbf{k}(X)/\mathbf{k}, \boldsymbol{\mu}_n).$$

But the action of a constant map factors through $H^2_{\mathrm{nr}}(\mathbf{k}/\mathbf{k}, \boldsymbol{\mu}_n)$, via the corestriction map, whence the result follows. $\square$

3 The deformation method

In this section, we describe the deformation method which produces irrational varieties. We show that if we have a good family of varieties, and if one of them does not have a decomposition of the diagonal, then a very general one in the family will not have a decomposition, and hence, will not be retract rational.

This method was developed by C. Voisin [Voi15], and modified by J.-L. Colliot-Thélène and A. Pirutka [CTP16], to show that a very general quartic threefold in $\mathbf{P}^4_{\mathbf{C}}$ is not retract rational. We will present a proof of this result.

Families of cycles

Definition 3.1 (Kollár [Kol96, Definition 3.10])**.** Suppose that

- **k** is an algebraically closed field of characteristic 0.
- S is a **k**-scheme.
- X/S is a projective S-scheme, with a chosen relatively ample line bundle.
- B/S is a reduced normal S-scheme.
- d and d' are non-negative integers.

A *well-defined family of d-cycles* of X of degree d' parametrised by B is a cycle

$$C = \sum_i m_i[C_i] \quad \text{of} \quad X \times_S B,$$

such that

- Each C_i is an integral closed subscheme of $X \times_S B$.
- For each i, the image of the projection map $g_i : C_i \to B$ is an irreducible component of B. In particular, g_i is flat over a dense open subset of B.
- Each fibre of g_i defines a d-cycle of X of degree d'. This means that the fibre is either empty or of dimension d, and that g_i is flat over a dense open subset of B.

The deep theorem below shows the existence of a universal family of cycles, in that every family of cycles is realised as its pullback.

Theorem 3.2. *Under the assumptions of Definition* 3.1*, for an S-scheme Z, define*

$$\mathrm{Chow}^{d,d'}_{X/S}(Z) = \left\{ \begin{array}{c} \text{well-defined families of non-negative } d\text{-cycles} \\ \text{of } X \text{ of degree } d' \text{ parametrised by } Z \end{array} \right\}.$$

Then

- $\mathrm{Chow}^{d,d'}_{X/S}$ *is a contravariant functor from the semi-normal S-schemes to sets.*
- *Moreover, this functor is represented by a projective semi-normal S-scheme* $\mathsf{Chow}^{d,d'}_{X/S}$*, called the* Chow scheme*, so that there exists a universal well-defined family of non-negative d-cycles*

$$\mathsf{Univ}^{d,d'}_{X/S} \quad \text{of } X \text{ parametrised by} \quad \mathsf{Chow}^{d,d'}_{X/S},$$

  such that every other family of cycles is its pullback.

See [Kol96, Theorem I.3.21].

We also recall the existence of Hilbert schemes.

Theorem 3.3. *Let S be a locally noetherian scheme. Let $X \to S$ be a projective morphism. For an S-scheme Z, define*

$$\mathrm{Hilb}^{d,d'}_{X/S}(Z) = \left\{ \begin{array}{c} \textit{closed subschemes of } X \times_S Z \textit{ flat over } Z \\ \textit{of relative dimension } d \textit{ and relative degree } d' \end{array} \right\}.$$

The functor $\mathrm{Hilb}^{d,d'}_{X/S}$ *is represented by an S-scheme* $\mathsf{Hilb}^{d,d'}_{X/S}$, *called the* Hilbert scheme, *whose irreducible components are projective over S. As a result, there exists a universal family of subschemes*

$$U \quad \subset \quad X \times_S \mathsf{Hilb}^{d,d'}_{X/S},$$

such that every other family of subschemes is its pullback.

Below, we will write

$$\mathsf{Chow}_{X/S} = \coprod_{d,d'} \mathsf{Chow}^{d,d'}_{X/S} \quad \text{and} \quad \mathsf{Hilb}_{X/S} = \coprod_{d,d'} \mathsf{Hilb}^{d,d'}_{X/S}.$$

Locus of rational equivalence

Situation 3.4. *Suppose*

- $\mathbf{k}$ *is an algebraically closed field of characteristic* 0.
- *B is a smooth* $\mathbf{k}$*-scheme.*
- *$X \to B$ is a projective morphism.*

Lemma 3.5. *In Situation* 3.4, *for any non-negative integer d, there exists*

- *A countable family of normal, irreducible, quasi-projective B-schemes $\{T_i\}$.*
- *For each index i, a family of smooth $(d+1)$-dimensional varieties $W_i \to T_i$, with two families of divisors $E_{i,1}, E_{i,2} \to T_i$ of W_i,*

such that

- *For any $b \in B$ and any subvariety $V \subset X_b$ of dimension $d+1$, there exists a desingularisation $\widetilde{V}$, such that for any two effective divisors D_1, D_2 of $\widetilde{V}$, such that $D_1 - D_2$ is principal, there exists i and $t \in (T_i)_b(\mathbf{k})$, such that the data $(\widetilde{V}, D_1, D_2)$ is identical to $((W_i)_t, (E_{i,1})_t, (E_{i,2})_t)$.*

The reason to consider a desingularisation of V, instead of V itself, is that on a smooth variety, a Weil divisor is the same thing as a Cartier divisor, and the Weil divisor class group is the same as the Picard group. The normality of T_i is required in order to (later) satisfy the definition of a well-defined family of cycles.

Proof. By [EGA-IV3, Theorem 9.7.7], the set of points in the Hilbert scheme $\mathrm{Hilb}^{d+1}_{X/B}$ corresponding to the geometrically integral subvarieties is locally constructible. Let G be an irreducible component of this set, equipped with the reduced scheme structure. Then G is quasi-projective over S, as the components of $\mathrm{Hilb}^{d+1}_{X/B}$ are projective. Let

$$W \subset G \times_B X$$

be the universal family of $(d+1)$-dimensional subschemes. The morphism $W \to G$ is thus projective, flat, with geometrically integral fibres.

The generic fibre $W_{\mathbf{k}(G)}$ is integral, as its irreducible components correspond to irreducible components of a general fibre. By Hironaka's theorem, let $\widetilde{W}_{\mathbf{k}(G)} \to W_{\mathbf{k}(G)}$ be a desingularisation map. This map extends to a map

$$\widetilde{W}_1 \to W_1$$

of schemes over an open set $G_1 \subset G$, where $W_1 = W|_{G_1}$. Shrinking G_1 if necessary, we can assume that for any $t \in G_1(\mathbf{k})$, the map $\widetilde{W}_{1,t} \to W_t$ of fibres over t is a desingularisation map.

By noetherian induction, we can find a decomposition

$$G = \textstyle\bigcup_{j=1}^m G_j,$$

with G_j locally closed in G, together with maps $\widetilde{W}_j \to W_j$ over G_j, where $W_j = W|_{G_j}$, such that for all $t \in G_j$, the map $\widetilde{W}_{j,t} \to W_t$ is a desingularisation map.

Let $\widetilde{G}_j \to G_j$ be a desingularisation, and we still denote by $\widetilde{W}_j \to \widetilde{G}_j$ the pullback of the family $\widetilde{W}_j \to G_j$.

Since $\widetilde{W}_j$ is projective and flat over $\widetilde{G}_j$, with geometrically integral fibres, there exist the schemes with a morphism

$$\mathrm{Ab} : \mathrm{Div}_{\widetilde{W}_j/\widetilde{G}_j} \to \mathrm{Pic}_{\widetilde{W}_j/\widetilde{G}_j},$$

where $\mathrm{Div}_{\widetilde{W}_j/\widetilde{G}_j}$ is the scheme parametrising the effective Cartier divisors [FAG, Theorem 9.3.7], which is quasi-projective over $\widetilde{G_j}$, and hence over B, and $\mathrm{Pic}_{\widetilde{W}_j/\widetilde{G}_j}$ is the Picard scheme [FAG, Theorem 9.4.8]. Let

$$\Delta_j \subset \mathrm{Div}_{\widetilde{W}_j/\widetilde{G}_j} \times \mathrm{Div}_{\widetilde{W}_j/\widetilde{G}_j}$$

be the inverse image of the diagonal of $\mathrm{Pic}_{\widetilde{W}_j/\widetilde{G}_j} \times \mathrm{Pic}_{\widetilde{W}_j/\widetilde{G}_j}$, under the map $\mathrm{Ab} \times \mathrm{Ab}$, equipped with the reduced scheme structure. Let T be one of its irreducible components, and let $\widetilde{T}$ be the normalisation of T. Thus $\widetilde{T}$ is quasi-projective over B.

The family of all the schemes $\widetilde{T}$, together with the two universal families of divisors given by the Div schemes, satisfies the requirement of the lemma. □

Of course, a rational equivalence of two d-cycles may involve more than one $(d+1)$-dimensional subvariety. The next lemma deals with this situation.

For simplicity, if $V \subset X_b$ is a subvariety, we will say "the desingularisation" of V when we refer to the variety $\widetilde{V}$ given by the previous lemma, and we simply add a tilde to indicate this desingularisation.

Lemma 3.6. *In Situation 3.4, for any non-negative integer d, there exists*

- *A countable family of normal irreducible B-schemes $\{H_i\}$.*
- *For each index i, an integer $n_i \geq 1$, and n_i triples $(W_{i,j},\ E_{i,j,1},\ E_{i,j,2})_{j=1}^{n_i}$, where for each j, $W_{i,j} \to H_i$ is a smooth projective family of $(d+1)$-dimensional varieties, and $E_{i,j,1},\ E_{i,j,2} \to H_i$ are two families of divisors of $W_{i,j}$,*

such that

- *For any $b \in B(\mathbf{k})$, and any data $(V_j,\ D_{j,1},\ D_{j,2})_{j=1}^{n}$, where each V_j is an integral subscheme of X_b of dimension $d+1$, and $D_{j,1},\ D_{j,2}$ are two effective Weil divisors on the desingularisation $\widetilde{V}_j$ of V_j, such that $D_{j,1} - D_{j,2}$ is a principal divisor on V_j, there exists i and $t \in (H_i)_b(\mathbf{k})$, such that the fibre $\big((W_i)_t,\ (E_{i,j,1})_t,\ (E_{i,j,2})_t\big)$ is identical to the given data.*

Proof. For each n-tuple $(T_1, \dots, T_n)$ as given by the previous lemma, we consider the normalisation H of the product $T_1 \times \cdots \times T_n$, equipped with the data of n triples as given by the previous lemma. The collection of all such H satisfies the requirement of this lemma. □

Our effort to parametrise all possibilities for a rational equivalence allows us to prove the following result.

Lemma 3.7. *In Situation 3.4, let*

$$Z_1, Z_2 \in \mathrm{Chow}^d_{X/B}(B)$$

be two well-defined families of cycles. Then there exists

- *A countable family of quasi-projective B-schemes $\{M_i\}$.*
- *For each index i, the data $(W_{i,j},\ E_{i,j,1},\ E_{i,j,2})_{j=1}^{n_i}$ as in Lemma 3.6, with M_i in place of H_i,*

such that

- *The union of the images of $M_i(\mathbf{k})$ in $B(\mathbf{k})$ is exactly the set*

$$\{b \in B(\mathbf{k}) \mid [Z_{1,b}] = [Z_{2,b}] \text{ in } \mathrm{CH}_\bullet(X_b)\}.$$

- *For any $b \in B(\mathbf{k})$, and any data $(V_i,\ D_{i,1},\ D_{i,2})_{i=1}^n$ as in Lemma 3.6, such that*
$$Z_{1,b} + \sum_{i=1}^n [D_{i,1}] = Z_{2,b} + \sum_{i=1}^n [D_{i,2}] \quad \text{in } \mathrm{Z}_\bullet(X_b),$$
there exists i and a point $t \in (M_i)_b(\mathbf{k})$, such that the fibre $\big((W_i)_t,\ (E_{i,j,1})_t,\ (E_{i,j,2})_t\big)$ is identical to the given data.

Proof. Let $\{H_i\}$ be the family in Lemma 3.6. We define a morphism
$$f : H_i \to \mathsf{Chow}_{X/B} \times \mathsf{Chow}_{X/B},$$
$$t \mapsto \Big(Z_1 + \sum_{j=1}^{n_i}(E_{i,j,1})_t,\ Z_2 + \sum_{j=1}^{n_i}(E_{i,j,2})_t\Big),$$
where $(E_{i,j,1 \text{ or } 2})_t$ is regarded as a cycle of X via the pushforward along the desingularisation map (onto a closed subvariety of X). Now let M_i be the inverse image of the diagonal along f, with the reduced scheme structure, equipped with the data of n_i triples given by that of H_i. This proves the second statement.

For the first statement, write $Z = Z_1 - Z_2$. Let $b \in B$ be a point where Z_b is rationally equivalent to zero. Then, there exist subvarieties $V_j \subset X_b$, where $j = 1, \dots, n$, and rational functions g_j on V_j, which give rise to rational functions $\widetilde{g}_j$ on a desingularisation $\widetilde{V}_j$, such that
$$Z_b = \sum_{j=1}^n (f_j)_*(\operatorname{div} \widetilde{g}_j),$$
where f_j denotes the map $\widetilde{V}_j \to X_b$. Conversely, the existence of this data implies that Z_b is rationally equivalent to zero, since M_j is taken to be the inverse image of the diagonal. Therefore, the locus where Z_b is equivalent to zero is exactly the union of the images of the M_i. □

We are now getting close to the main theorem, which states that the locus where $[Z_{1,b}] = [Z_{2,b}]$ is a countable union of closed sets. There is one further lemma needed.

Lemma 3.8. *Let M be a smooth $\mathbf{k}$-variety of dimension m, with $\mathbf{k}$ algebraically closed, and let $f : W \to M$ be a flat morphism of relative dimension r. Let Z be an n-cycle on W. Suppose that*

- *There is a dense open set $M^\circ \subset M$, such that $Z|_{f^{-1}(M^\circ)}$ is rationally equivalent to 0 in $f^{-1}(M^\circ)$.*

Then

- *For any $t \in M(\mathbf{k})$, the fibre Z_t is rationally equivalent to 0 in W_t.*

Proof. Let $t \in M(\mathbf{k}) \setminus M^\circ(\mathbf{k})$ be a point. As in the proof of Lemma 2.9, we can find a curve C in M, passing through t, and not contained in $M \setminus M^\circ$. Taking the normalisation of this curve, we may thus assume that M is a smooth curve.

Let $D = M \setminus M^\circ$, which is now a finite set. There is an exact sequence [Ful98, §1.8]

$$\mathrm{CH}_n(f^{-1}(D)) \xrightarrow{i_*} \mathrm{CH}_n(W) \longrightarrow \mathrm{CH}_n(f^{-1}(M^\circ)) \to 0,$$

so that $Z = i_*(z)$ for some $z \in \mathrm{CH}_n(f^{-1}(D))$, where $i: f^{-1}(D) \to W$ denotes the inclusion. But by the projection formula [Ful98, §2.3], the intersection of $i_*(z)$ with the divisor $f^{-1}(D)$ of W is $i_* i^*(f^{-1}(D) \cdot z) = 0$, so that Z_t is rationally equivalent to 0 for any $t \in D$. □

Now we are ready to prove the main result, and our proof follows that of [Voi15, Proposition 2.4].

Theorem 3.9 (Voisin). *In Situation* 3.4*, let*

$$Z_1, Z_2 \in \mathrm{Chow}^d_{X/B}(B)$$

be two well-defined families of cycles. Then there exists a countable family $\{B_i\}$ of closed subschemes of B, such that

$$\left\{ b \in B(\mathbf{k}) \;\middle|\; [Z_{1,b}] = [Z_{2,b}] \text{ in } \mathrm{CH}_d(X_b) \right\} = \textstyle\bigcup_i B_i(\mathbf{k}).$$

Proof. Let $\{M_i\}$ be as in Lemma 3.7. Replacing each M_i by its desingularisation, we can assume that all M_i are smooth.

Let $B_i \subset B$ be the closure of the image of M_i in B, as a closed integral subvariety. By Lemma 3.7, the equation $[Z_{1,b}] = [Z_{2,b}]$ implies $b \in B_i$ for some i. Thus, it suffices to show that it holds for all $b \in B_i$.

Let $B_i^\circ \subset B_i$ be an open subset contained in the image of M_i, and let M_i° be the inverse image of B_i° in M_i. Let $X_{M_i} = X \times_B M_i$, and Z_i the pullback of $Z = Z_1 - Z_2$ along the morphism $X_{M_i} \to X$, which is actually the Chow pullback of families of cycles along the map $M_i \to B$. Then Z_i is equal to the universal cycle $\sum_{j=1}^{n_i} (E_{i,j,1} - E_{i,j,2})$ on M_i, and hence is rationally equivalent to zero.

By taking the closure in a projective bundle, the morphism $M_i \to B_i$ extends to a projective morphism $\overline{M}_i \to B_i$. Again, taking a desingularisation, we may assume $\overline{M}_i$ is smooth. Write $X_{\overline{M}_i} = X \times_B \overline{M}_i$. Now apply Lemma 3.8 with $M = \overline{M}_i$, $W = X_{\overline{M}_i}$, and M° the inverse image of M_i°. This shows that for all $b \in B_i$, the cycle Z_b is equivalent to zero. □

Locus of decomposability of the diagonal

Lemma 3.10. *Suppose*

- $\mathbf{k}$ *is an algebraically closed field of characteristic* 0.
- B *is a smooth* $\mathbf{k}$*-scheme.*
- $X \to B$ *is a projective morphism, and write* $Y = X \times_B X$.

Then there exists

- *A countable family of smooth irreducible B-schemes $\{F_i\}$.*
- *For each index i, a well-defined family of non-negative d_i-cycles C_i of Y of degree d_i', parametrised by F_i,*

such that

- *For any $b \in B$ and any non-negative d-cycle C of Y_b of degree d', supported in $Z \times X_b$ for a codimension 1 subset $Z \subset X_b$, there exists i and $x \in (F_i)_b$ such that $C = (C_i)_x$.*
- *For any $x \in (F_i)_b$, the cycle $C = (C_i)_x$ is supported in $Z \times X_b$ for a codimension 1 subset $Z \subset X_b$.*

The condition "supported in $Z \times X_b$" is the main point of this lemma. In fact, the proof would be a lot easier if we dropped this condition. This lemma will be used to parametrise all possibilities for the term D in a decomposition of the diagonal, as in Definition 2.14.

Proof. First, we need to parametrise all the subschemes of X that are codimension 1 in X_b at each $b \in B$. Therefore, we consider an irreducible component

$$H \subset \mathsf{Hilb}_{X/B},$$

parametrising the codimension 1 subschemes. Let $U \subset H \times_B X$ be the universal subscheme. Thus if we look at the fibre at $b \in B$, then H_b parametrises the codimension 1 subschemes of X_b, and $U_b \subset H_b \times X_b$ is a subscheme whose intersection with $\{c\} \times X_b$ gives the subscheme of X_b corresponding to c.

Next, we want to parametrise all the subschemes of Y which have the form (codim 1 subset) $\times X_b$ when restricted to the fibres. This is given by the universal subscheme

$$U' = U \times_B X \subset H \times_B X \times_B X,$$

which, at $b \in B$, when intersected with $\{c\} \times X_b \times X_b$, gives the subscheme of $X_b \times X_b$ corresponding to c.

Finally, we parametrise cycles of Y supported in a subset of the form of the previous step. Thus we consider an irreducible component

$$C \subset \mathsf{Chow}_{U'/H}.$$

Let $V \in \mathrm{Z}_\bullet(C \times_H U')$ be the universal family. Since

$$C \times_H U' \subset C \times_H H \times_B X \times_B X \simeq C \times_B X \times_B X,$$

we can view V as a family of cycles of Y parametrised by C.

Thus, all choices of H and C will give a countable set of families, which together parametrise all the cycles of Y of the given form.

However, the parametrising schemes need to be smooth. We thus apply Hironaka's desingularisation theorem to the schemes C. □

Proposition 3.11. *Suppose*

- $\mathbf{k}$ *is an algebraically closed field of characteristic* 0.
- B *is a smooth* $\mathbf{k}$*-scheme.*
- $X \to B$ *is a projective morphism.*

Then there exists a countable family $\{B_i\}$ *of closed subschemes of* B*, such that*

$$\{b \in B(\mathbf{k}) \mid X_b \text{ has a decomposition of the diagonal}\} = \bigcup_i B_i(\mathbf{k}).$$

Proof. Let $F_i, F_{i'}$ be two of the schemes as in Lemma 3.10, with $d_i = d_{i'} = \dim(X/B)$, and let $C_i, C_{i'}$ be the universal cycles, lying in $X \times_B X \times_B F_{i \text{ or } i'}$.

Let $G_j, G_{j'}$ be irreducible components of $\mathrm{Chow}^{0,d}_{X/B}$ and $\mathrm{Chow}^{0,d+1}_{X/B}$, respectively, where d is arbitrary, and let $D_j, D_{j'}$ be the universal cycles lying in $X \times_B G_{j \text{ or } j'}$.

We define two cycles of $Y = F_i \times_B G_j \times_B X \times_B X \times_B G_{j'} \times_B F_{i'}$ by

$$\begin{aligned} Z_1 &= \Big([G_j] \times C_i + [F_i] \times [G_j] \times [\Delta_{X/B}] + [F_i] \times [X] \times D_j\Big) \times [G_{j'}] \times [F_{i'}], \\ Z_2 &= [F_i] \times [G_j] \times \Big(C_{i'} \times [G_{j'}] + [X] \times D_{j'} \times [F_{i'}]\Big), \end{aligned}$$

where $[\Delta_{X/B}]$ is the diagonal class.

Now apply Theorem 3.9, where we take X to be Y, and take B to be

$$F_i \times_B F_{i'} \times_B G_j \times_B G_{j'}.$$

At the point

$$t = (t_1, t_2, x_1, x_2) \in (F_i)_b \times (F_{i'})_b \times (G_j)_b \times (G_{j'})_b,$$

the cycle Z_1 gives $[\Delta_{X_b}] + z_1 + [X_b] \times x_1$, where z_1 is a non-negative cycle supported in $Z \times X_b$ for $Z \subset X_b$ of codimension 1, and similarly, the cycle Z_2 gives $[X_b] \times x_2 + z_2$, with z_2 likewise.

Therefore, Theorem 3.9 implies that the locus where the equation

$$[\Delta_{X_b}] + z_1 + [X_b] \times x_1 = [X_b] \times x_2 + z_2 \quad \in \mathrm{CH}_{\dim X_b}(X_b \times X_b)$$

holds (for non-negative x_1, x_2, z_1, z_2) is the union of countably many closed subsets. □

This result is restated as follows.

Theorem 3.12. *Suppose*

- $\mathbf{k}$ *is an algebraically closed field of characteristic* 0.
- B *is a smooth* $\mathbf{k}$*-scheme.*

- *$X \to B$ is a dominant projective morphism.*
- *There exists a* $\mathbf{k}$*-point* $0 \in B$*, such that the fibre* X_0 *does not have a decomposition of the diagonal.*

Then for a "very general" $\mathbf{k}$*-point* $b \in B$*, the fibre* X_b *will not have a decomposition of the diagonal.* □

By "*very general*", we mean "except a countable union of closed sets of codimension ≥ 1".

This means that if we can find one example in a family of varieties, which we can show has non-trivial Brauer group, and hence does not have a decomposition of the diagonal, then a very general variety in this family is not retract rational.

Stable equivalence

This subsection gives a variant of the above result, concerning stable equivalence instead of retract rationality.

Definition 3.13. Two projective $\mathbf{k}$-varieties are *stably equivalent*, if

$$X \times \mathbf{P}^m \quad \text{is birational to} \quad Y \times \mathbf{P}^n$$

for some $m, n \in \mathbf{N}$.

Stable rationality is the same as stable equivalence to a point.

Lemma 3.14. *Let* X, Y *be two* $\mathbf{k}$*-varieties, such that there exist open sets* $U \subset X$*,* $V \subset Y \times \mathbf{P}^n$*, and two morphisms* $p : U \to V$*,* $q : V \to U$*, such that* $q \circ p = \mathrm{id}_U$*. Then there exist two correspondences*

$$f \in \mathrm{Corr}(X, Y), \quad g \in \mathrm{Corr}(Y, X),$$

such that for any field extension $\mathbf{K}/\mathbf{k}$*, the induced map*

$$(g \circ f)_* : \mathrm{CH}_0(X_{\mathbf{K}}) \to \mathrm{CH}_0(X_{\mathbf{K}})$$

is the identity map. When X *is smooth, we have a decomposition*

$$[\Delta_X] = D + g \circ f \quad \textit{in } \mathrm{Corr}(X, X),$$

where D *is supported in* $Z \times X$ *for some closed subvariety* $Z \subset X$ *of codimension at least* 1.

Proof. The correspondence f is given by the rational map

$$X \overset{\supset}{\dashrightarrow} U \overset{p}{\to} V \hookrightarrow Y \times \mathbf{P}^n \to Y,$$

and g is given by a rational map

$$Y \hookrightarrow Y \times \mathbf{P}^n \overset{\supset}{\dashrightarrow} V \overset{q}{\to} U \hookrightarrow X,$$

where the inclusion $Y \hookrightarrow Y \times \mathbf{P}^n$ is chosen so that the composition is defined. To prove that $g \circ f$ induces the identity map on $\mathrm{CH}_0(X_\mathbf{K})$, it suffices to prove that the map

$$Y \times \mathbf{P}^n \to Y \hookrightarrow Y \times \mathbf{P}^n$$

induces the identity map on CH_0. This is because every closed point of $Y \times \mathbf{P}^n$ is sent to another point that lives in the same slice of $\mathbf{P}^n$, and hence, is rationally equivalent to it as a 0-cycle.

For the second part, we use an argument as in the proof of Theorem 2.18. Namely, we change the base field to $\mathbf{k}(X)$, to find that

$$g_{\mathbf{k}(X)} \circ f_{\mathbf{k}(X)}(\beta) = \beta \quad \text{in } \mathrm{CH}_0(X_{\mathbf{k}(X)}),$$

where β is the class of the generic point. The rest of the proof is analogous to the proof of Theorem 2.18, (ii) $\Rightarrow$ (iii). □

Note that the assumptions of this lemma is satisfied when X and Y are stably equivalent.

Using an argument as in the proof of Proposition 3.11, we obtain the following result.

Theorem 3.15. *Suppose*

- $\mathbf{k}$ *is an algebraically closed field of characteristic* 0.
- B *is a smooth* $\mathbf{k}$*-scheme.*
- $X \to B$ *and* $Y \to B$ *are two projective morphisms.*

Then the set of all points $b \in B(\mathbf{k})$ *such that there exist correspondences*

$$f \in \mathrm{Corr}(X_b, Y_b), \quad g \in \mathrm{Corr}(Y_b, X_b),$$

and D *as before, such that*

$$[\Delta_{X_b}] = D + g \circ f,$$

is a countable union of closed sets.

Proof. We apply Theorem 3.9, where we take X to be

$$X \times_B X \times_B F_i \times_B F_{i'} \times_B G_j \times_B G_{j'} \times_B H_k \times_B H_{k'},$$

where

- F_i, $F_{i'}$ are given by Lemma 3.5.
- $G_j, G_{j'}, H_k, H_{k'}$ are irreducible components of $\mathsf{Chow}_{X \times_B Y/B}$, parametrising the correspondences from X to Y for G_j and $G_{j'}$, and from Y to X for H_k, $H_{k'}$.

The rest of the proof is analogous to the proof of Proposition 3.11. □

Corollary 3.16. *Under the assumptions of Theorem 3.15, the set of all points $b \in B(\mathbf{k})$ such that X_b is stably equivalent to Y_b is contained in a countable union of closed sets. Moreover, this union does not contain any point $b \in B(\mathbf{k})$ such that X_b is smooth and has a decomposition of the diagonal, and Y_b does not have a decomposition of the diagonal*

Proof. The countable union of closed sets given by Theorem 3.15 satisfies this requirement. Indeed, for those $b \in B(\mathbf{k})$ such that X_b and Y_b are stably equivalent, Lemma 3.14 shows that b is in this union.

To prove the last statement, let $b \in B(\mathbf{k})$ be such a point. We show that such correspondences f, g as in Theorem 3.15 do not exist between X_b and Y_b. In fact, if they exist, then $g \circ f$ acts on $\mathrm{CH}_0(X)$ by the identity map. But since id_Y sends every 0-cycle to its degree multiplied by a fixed 0-cycle of degree 1, so does the correspondence $g \circ f = g \circ \mathrm{id}_Y \circ f$. Moreover, this holds over any field extension of $\mathbf{k}$. By Theorem 2.18, X has a decomposition of the diagonal, a contradiction. □

In particular, if we take X to be a constant family which is smooth, we deduce that every stable equivalence class in a family of varieties is contained in a countable union of closed sets.

Corollary 3.17. *Suppose*

- $\mathbf{k}$ *is an uncountable algebraically closed field of characteristic* 0.
- B *is a smooth* $\mathbf{k}$*-scheme.*
- $X \to B$ *is a dominant projective morphism, with smooth generic fibre.*
- *There exist two* $\mathbf{k}$*-points $b_0, b_1 \in B$, such that the fibre X_{b_0} has a decomposition of the diagonal, while the fibre X_{b_1} does not have a decomposition of the diagonal.*

Then there are uncountably many stable equivalence classes of varieties in this family.

Proof. For those smooth fibres X_b that do not have a decomposition of the diagonal, we apply Corollary 3.16 to the constant family $B \times X_b \to B$ and the family $X \to B$. It follows that the set of $b' \in B(\mathbf{k})$ such that X_b is stably equivalent to $X_{b'}$ is contained in a countable union of closed sets, which can not coincide with the whole space.

By Theorem 3.12, the locus of smooth fibres with no decomposition of the diagonal is the complement of a countable union of closed subsets of B. Therefore, in order to cover this locus, there must be uncountably many stable equivalence classes of fibres of $X \to B$, since each of these classes is contained in a countable union of closed sets. □

4 The specialisation method

The specialisation map

The main idea of the specialisation method is to build a way to transport properties between the generic fibre and the special fibre. We consider the following situation.

Situation 4.1. *Let A be a discrete valuation ring, with fraction field* $\mathbf{K}$ *and residue field* $\mathbf{k}$*. Let $\mathcal{X}$ be an A-scheme. Suppose that*

- *The special fibre $X^{\mathrm{s}} = \mathcal{X} \times_A \mathbf{k}$ is a $\mathbf{k}$-variety.*
- *The generic fibre $X = \mathcal{X} \times_A \mathbf{K}$ is a $\mathbf{K}$-variety.*

After Colliot-Thélène and Pirutka [CTP16], we introduce the specialisation map on the Chow groups.

Proposition 4.2. *In Situation* 4.1*, there is a* specialisation map

$$\sigma : \mathrm{CH}_0(X) \to \mathrm{CH}_0(X^{\mathrm{s}}),$$

which preserves the degree of 0*-cycles.*

Proof. By [Ful98, §1.8 and §20.1], there is an exact sequence

$$\mathrm{CH}_1(X^{\mathrm{s}}) \xrightarrow{i_*} \mathrm{CH}_1(\mathcal{X}) \xrightarrow{j^*} \mathrm{CH}_0(X) \to 0,$$

where i, j are the obvious inclusions. (The last term is CH_0 instead of CH_1, since $\operatorname{Spec} \mathbf{K}$ is a 1-dimensional point in $\operatorname{Spec} A$.)

By [Ful98, §2.6 and §20.1], there is a Gysin map

$$i^! : \mathrm{CH}_1(\mathcal{X}) \to \mathrm{CH}_0(X^{\mathrm{s}}),$$

given by intersection with the divisor X^{s} of $\mathcal{X}$. By [Ful98, Proposition 2.6 (c)], we have $i^! \circ i_* = 0$. Thus the map $i^!$ factors through the cokernel of i_*, giving the desired map. □

Lemma 4.3. *In Situation* 4.1*, suppose that A is henselian, and $\mathcal{X}$ is proper and flat over A. Let $X^{\mathrm{s}}_{\mathrm{sm}} \subset X^{\mathrm{s}}$ be the open set where X^{s} is smooth. Then every* 0*-cycle of X^{s} supported in $X^{\mathrm{s}}_{\mathrm{sm}}$ can be lifted along the specialisation map*

$$\sigma : \mathrm{CH}_0(X) \to \mathrm{CH}_0(X^{\mathrm{s}})$$

to a 0*-cycle supported in X_{sm}.*

Proof. We follow [EKW16, §4]. It is enough to lift the closed points of $X^{\mathrm{s}}_{\mathrm{sm}}$. Let $x \in X^{\mathrm{s}}_{\mathrm{sm}}$ be a closed point, and let $a_1, \dots, a_n \in \mathcal{O}_{X^{\mathrm{s}},x}$ be a regular sequence generating the maximal ideal. Choose liftings $\bar{a}_1, \dots, \bar{a}_n \in \mathcal{O}_{\mathcal{X},x}$. Since $\mathcal{O}_{X^{\mathrm{s}},x} \simeq$

$\mathcal{O}_{\mathcal{X},x}/\pi\mathcal{O}_{\mathcal{X},x}$, where $\pi \in A$ is a uniformiser, it follows that $\pi, \bar{a}_1, \dots, \bar{a}_n$ is a regular sequence in the $(n+1)$-dimensional local ring $\mathcal{O}_{\mathcal{X},x}$. Therefore, the ideal $(\bar{a}_1, \dots, \bar{a}_n) \subset \mathcal{O}_{\mathcal{X},x}$ defines a 1-dimensional subset of Spec $\mathcal{O}_{\mathcal{X},x}$, whose closure in $\mathcal{X}$ is a 1-dimensional subscheme $Z \subset \mathcal{X}$. Then Z is flat of relative dimension 0 over A, and hence quasi-finite over A. By properness, it is finite over A. It follows that $Z \simeq \operatorname{Spec} B$ for a finite A-algebra B. Since A is henselian, B is a product of local rings. Therefore, the irreducible component of Z containing x meets X^{s} at a single point x. The corresponding 0-cycle of X has the desired property. □

Lemma 4.4. *In Situation* 4.1*, suppose that*

- *A is henselian, and $\mathcal{X}$ is proper and flat over A.*
- *The generic fibre X has a desingularisation $p\colon \widetilde{X} \to X$, such that $\widetilde{X}$ is universally* CH_0*-trivial.*

Then every 0*-cycle of X^{s} of degree* 0*, supported in the open set $X^{\mathrm{s}}_{\mathrm{sm}} \subset X^{\mathrm{s}}$ where X^{s} is smooth, is zero in* $\mathrm{CH}_0(X^{\mathrm{s}})$.

Proof. Let $U \subset X$ be a dense open set such that $p\colon p^{-1}(U) \to U$ is an isomorphism. Let x be a 0-cycle of $X^{\mathrm{s}}_{\mathrm{sm}}$ of degree 0. By Lemma 4.3, x lifts to a 0-cycle of X_{sm} of degree 0. By the moving lemma 2.9, it is equivalent to a 0-cycle supported in U, which then lifts to a 0-cycle of $\widetilde{X}$. This 0-cycle is equivalent to 0 in $\widetilde{X}$ by hypothesis. Therefore, applying the map

$$\mathrm{CH}_0(\widetilde{X}) \xrightarrow{p_*} \mathrm{CH}_0(X) \xrightarrow{\sigma} \mathrm{CH}_0(X^{\mathrm{s}}),$$

we see that $x = 0$ in $\mathrm{CH}_0(X^{\mathrm{s}})$. □

Rationality and specialisation

The main result is that the rationality (or more precisely, universal CH_0-triviality) of the generic fibre can be specialised to the special fibre, so that once we show that the special fibre is irrational, we know that the generic fibre is also irrational.

First, we make clear what we need to obtain universal triviality for a desingularisation.

Lemma 4.5. *Let $f : \widetilde{X} \to X$ be a desingularisation map between two projective, geometrically integral* $\mathbf{k}$*-varieties. Suppose that*

- *$\widetilde{X}$ has a* 0*-cycle of degree* 1*.*
- *f is universally* CH_0*-trivial.*
- *There exists an open set $U \subset X$, with $\widetilde{U} = f^{-1}(U)$, such that $f : \widetilde{U} \to U$ is an isomorphism, and for any extension $F/\mathbf{k}$, every* 0*-cycle of degree* 0 *supported in U_F is rationally equivalent to zero in X_F.*

Then $\widetilde{X}$ is universally CH_0*-trivial.*

Proof. Since the conditions of the lemma is preserved by a base change, we only need to prove that $\deg_{\mathbf{k}} : \mathrm{CH}_0(\widetilde{X}) \to \mathbf{Z}$ is an isomorphism. By the first hypothesis, this map is surjective. Thus, it suffices to show that for every 0-cycle x of $\widetilde{X}$ of degree 0, x is rationally equivalent to 0. But by the moving lemma 2.9, it is equivalent to one supported in $\widetilde{U}$, which induces a cycle in U, which is equivalent to 0 in X by hypothesis. Since f is universally CH_0-trivial, x is equivalent to 0 in $\widetilde{X}$. □

Before the main theorem, we mention a convenient result in commutative algebra.

Lemma 4.6. *Let A be a discrete valuation ring with residue field $\mathbf{k}$, and let $F/\mathbf{k}$ be an extension. Then there exists a complete discrete valuation ring B with residue field F, together with a local map $A \to B$ inducing the field map $\mathbf{k} \to F$.*

See [Bou06, Chapter IX, Appendix, §2, Corollary, and Exercise 4].

Theorem 4.7 (Colliot-Thélène and Pirutka)**.** *In Situation* 4.1*, suppose that*

- *$\mathscr{X}$ is faithfully flat and proper over A, with geometrically integral fibres.*
- *The special fibre X^{s} has a desingularisation $f : \widetilde{X}^{\mathrm{s}} \to X^{\mathrm{s}}$, such that f is universally* CH_0*-trivial, and $\widetilde{X}^{\mathrm{s}}$ has a* 0*-cycle of degree* 1*.*
- *The generic fibre X has a desingularisation $\widetilde{X} \to X$.*

Then if $\widetilde{X}$ is universally CH_0*-trivial, so is $\widetilde{X}^{\mathrm{s}}$.*

Proof. The proof is done by putting the previous lemmas together.

- By Theorem 2.18, it suffices to show that $\widetilde{X}^{\mathrm{s}}_{\mathbf{k}(X^{\mathrm{s}})}$ is CH_0-trivial, where we notice that $\mathbf{k}(\widetilde{X}^{\mathrm{s}}) \simeq \mathbf{k}(X^{\mathrm{s}})$.
- By Lemma 4.5, it suffices to show that the open set $U = (X^{\mathrm{s}}_{\mathbf{k}(X^{\mathrm{s}})})_{\mathrm{sm}} \subset X^{\mathrm{s}}_{\mathbf{k}(X^{\mathrm{s}})}$ satisfies the third assumption of Lemma 4.5.
- By Lemma 4.4, it suffices to show that $X^{\mathrm{s}}_{\mathbf{k}(X^{\mathrm{s}})}$ can act the rôle of X^{s} in that lemma.
- By Lemma 4.6, we take a complete discrete valuation ring B, with residue field $\mathbf{k}(X^{\mathrm{s}})$, and a local map $A \to B$ inducing the map of fields $\mathbf{k} \to \mathbf{k}(X^{\mathrm{s}})$. Then B is henselian. Doing a base change along the map $A \to B$ for everything will complete the proof. □

There is a stronger variant of this result, which considers the geometrical generic fibre over $\overline{\mathbf{K}}$, instead of over $\mathbf{K}$. Before introducing the result, we need a lemma.

Lemma 4.8. *Let X be a smooth, integral, projective* $\mathbf{k}$*-variety. If $X_{\overline{\mathbf{k}}}$ is universally* CH_0*-trivial, then X_F is universally* CH_0*-trivial for some finite extension $F/\mathbf{k}$.*

Proof. By Theorem 2.18, $X_{\overline{\mathbf{k}}}$ has a decomposition of the diagonal

$$[\Delta_{X_{\overline{\mathbf{k}}}}] = D_{\overline{\mathbf{k}}} + [X_{\overline{\mathbf{k}}}] \times x_0 \quad \text{in } \mathrm{CH}_n(X_{\overline{\mathbf{k}}} \times_{\overline{\mathbf{k}}} X_{\overline{\mathbf{k}}}).$$

By Galois descent, there exists a finite extension $F/\mathbf{k}$ over which everything in the equation is defined, and we have

$$\mathrm{CH}_n(X_{\overline{\mathbf{k}}} \times_{\overline{\mathbf{k}}} X_{\overline{\mathbf{k}}}) = \operatorname*{colim}_{E/F \text{ finite}} \mathrm{CH}_n(X_E \times_E X_E).$$

Therefore, there exists a finite extension E such that the equation of the decomposition of the diagonal holds over E. □

Theorem 4.9 (Colliot-Thélène and Pirutka)**.** *In Situation* 4.1*, suppose that*

- *The residue field* $\mathbf{k}$ *is algebraically closed.*
- $\mathcal{X}$ *is faithfully flat and proper over* A*, with geometrically integral fibres.*
- *The special fibre* X^{s} *has a desingularisation* $f : \widetilde{X}^{\mathrm{s}} \to X^{\mathrm{s}}$*, such that* f *is universally* CH_0*-trivial.*
- *The geometrical generic fibre* $\overline{X} = \mathcal{X} \times_A \overline{\mathbf{K}}$ *is a* $\overline{\mathbf{K}}$*-variety, with a desingularisation* $\widetilde{X} \to \overline{X}$*.*

Then if $\widetilde{X}$ *is universally* CH_0*-trivial, so is* $\widetilde{X}^{\mathrm{s}}$*.*

Proof. Our plan is to find a suitable base change in order to apply Theorem 4.7.

First, we replace A by its completion.

Let F be a finite extension of $\mathbf{K}$, on which $\widetilde{X}$ is defined. In other words, there exists a smooth variety Y over F, such that $Y_{\overline{\mathbf{K}}} \simeq \widetilde{X}$, and there is a desingularisation map $Y \to X_F$, which coincides with the map $\widetilde{X} \to \overline{X}$ over $\overline{\mathbf{K}}$.

By Lemma 4.8, we may replace F by a finite extension of it, so we may assume that Y is universally CH_0-trivial.

Let B be the integral closure of A in F. By [Ser79, Proposition I.3], B is also a discrete valuation ring. Since $\mathbf{k}$ is algebraically closed, the residue field of B is also $\mathbf{k}$. We can thus do a base change along the map $A \to B$, and apply Theorem 4.7 to complete the proof. □

There is an even stronger variant of this result, where instead of $\overline{\mathbf{K}}$, we consider any field containing $\overline{\mathbf{K}}$.

Lemma 4.10. *Let* X *be a projective* $\mathbf{k}$*-variety. If* X_F *is retract rational for some extension* $F/\mathbf{k}$*, then the same is true for a certain finite extension* $F/\mathbf{k}$*.*

Proof. In the definition of the retract rationality of X_F, everything is defined over a finitely generated extension of $\mathbf{k}$. Thus we may assume $F/\mathbf{k}$ is finitely generated.

For the same reason, the lemma is true when F is the algebraic closure of $\mathbf{k}$. Therefore, we may assume that $\mathbf{k}$ is algebraically closed.

In this case, there exists a $\mathbf{k}$-variety Y such that F is isomorphic to $\mathbf{k}(Y)$, and there exist non-empty open sets $U \subset X \times_{\mathbf{k}} Y$ and $V \subset \mathbf{P}^n_Y$, such that U is a retract of V as a Y-scheme. There exists a closed point $y \in Y$ such that the fibres U_y and V_y are non-empty. This proves the lemma. □

Theorem 4.11 (Colliot-Thélène and Pirutka)**.** *Assume that the four assumptions of Theorem* 4.9 *are satisfied. Then, if $\widetilde{X}_F$ is retract rational for a field F containing $\overline{\mathbf{K}}$, then $\widetilde{X}^{\mathrm{s}}$ is universally* CH_0*-trivial.* □

5 Example: Quartic threefolds

In this section, we construct an explicit example of a quartic threefold which is unirational, but has non-trivial Brauer group. This example was originally due to Artin and Mumford [AM72], and slightly modified in [CTP16] so that it embeds in $\mathbf{P}^4$.

The example

- Let $\mathbf{k}$ be an algebraically closed field with $\operatorname{char} \mathbf{k} \neq 2$.
- Let $A \subset \mathbf{P}^2$ be a smooth conic, defined by the quadratic equation

$$\alpha(z_0, z_1, z_2) = 0.$$

- Let $D_1, D_2 \subset \mathbf{P}^2$ be two smooth cubics, defined by

$$\delta_1 = 0 \quad \text{and} \quad \delta_2 = 0,$$

  such that they each meet A tangentially at 3 points, giving six tangent points $Q_1, \dots, Q_6$, and such that $D_1 \cap D_2$ is nine distinct points $P_1, \dots, P_9$. These nine points do not lie on A, since otherwise D_1 and D_2 will meet tangentially at that point.

- Let $B \subset \mathbf{P}^2$ be a cubic that intersects A in the six points $Q_1, \dots, Q_6$. In fact, for any nine given points on the plane, there exists a cubic curve passing through all of them. We use these six points, and choose three other points which are non-collinear and not on A. This ensures that the cubic does not contain A, and can only intersect A in these six points.

  As cycles of A, we have

$$(D_1 + D_2) \cdot A = 2B \cdot A.$$

  This means that $\alpha \mid \delta_1\delta_2 - \beta^2$, where β is the polynomial defining B. Thus we may write

$$\delta_1\delta_2 = \beta^2 - 4\alpha\gamma$$

  for some γ of degree 4.

- Let $S \subset \mathbf{P}^3$ be the quartic surface defined by

$$g = \alpha(z_0, z_1, z_2)\, z_3^2 + \beta(z_0, z_1, z_2)\, z_3 + \gamma(z_0, z_1, z_2) = 0.$$

  Using the projection to $\mathbf{P}^2$ which sends $(z_0 : z_1 : z_2 : z_3)$ to $(z_0 : z_1 : z_2)$, the surface $S \setminus (0 : 0 : 0 : 1)$ can be seen as a double cover of $\mathbf{P}^2$, ramified along the curves D_1 and D_2.

- After applying a linear coordinate change in z_0, z_1, z_2, we may assume

$$\begin{aligned}&\text{The hyperplane } z_0 = 0 \text{ does not contain } Q_1, \ldots, Q_6,\\ &\qquad\qquad\text{or any point of } M \setminus \{P_0\}, \text{ and is not tangent to } A, \end{aligned}\quad (5.0.1)$$

  where

$$M = \left\{ g = 0,\ \left(\frac{\partial g}{\partial z_1} = 0 \text{ or } \frac{\partial g}{\partial z_2} = 0\right),\ \frac{\partial g}{\partial z_3} = 0 \right\},$$

  and $P_0 = (0 : 0 : 0 : 1)$. This is a technical assumption which we make to avoid bad singularities.

- Now let $T \subset \mathbf{P}^4$ be the quartic threefold defined by

$$f = \alpha(z_0, z_1, z_2)\, z_3^2 + \beta(z_0, z_1, z_2)\, z_3 + \gamma(z_0, z_1, z_2) + z_0^2 z_4^2 = 0.$$

  It is a double cover of $\mathbf{P}^3$, ramified along the surface S and the hyperplane $z_0 = 0$.

We will see that T has the property of being unirational but not having a decomposition of the diagonal.

We next construct an explicit desingularisation of the threefold T constructed above, following [CTP16, Appendix A]. We do this in order to get a desingularisation map which is universally CH_0-trivial, and such that the Brauer group of the desingularisation is non-trivial.

- We observe that the threefold T is singular along the line

$$L :\ z_0 = z_1 = z_2 = 0$$

  in $\mathbf{P}^4$, and outside of this line, it has ordinary quadratic singularities at the points $P_1, \ldots, P_9$ on the hyperplane $z_4 = 0$.

- Let $T_1 \to T$ be the blow-up along L. Standard computation shows that the exceptional divisor is a rational surface, and T_1 is singular along a line L_1 which is the inverse image of the point $P = (0 : 0 : 0 : 0 : 1)$.

- Let $T_2 \to T_1$ be the blow-up along L_1. Standard computation shows that the exceptional divisor is a rational surface, and T_2 only has ordinary quadratic singularities, at the inverse images of the nine points $P_1, \ldots, P_9$.

- Finally, we blow up T_2 at the nine points $P_1, \dots, P_9$. This gives a desingularisation of T. The exceptional divisor over the nine points are rational surfaces, and over L, it is a union of two rational surfaces. We can thus apply Theorem 2.13 to conclude that the desingularisation map is universally CH_0-trivial.
- Artin and Mumford [AM72, §2] showed that over $\mathbf{C}$, there is a smooth projective threefold V, birational to T, such that the singular cohomology $H^3(V, \mathbf{Z})$ contains non-trivial 2-torsion. It follows from the universal coefficient theorem and the comparison theorem [Mil80, Theorem III.3.12, p. 117] that the étale cohomology group $H^3(V, \mathbf{Z}_2)$ contains non-trivial 2-torsion. By Lemma 5.4 below, $\mathrm{Br}(V)$ contains non-trivial 2-torsion, and so does $\mathrm{Br}(T_2)$ by Theorem 2.21. In particular, $\mathrm{Br}(T_2) \neq 0$. This shows that T is not retract rational, by Theorem 2.23.

  For general $\mathbf{k}$ of characteristic zero, a consequence of the smooth base change theorem for étale cohomology [Mil80, Corollary VI.4.3, p. 231] shows that $\mathrm{Br}(T_2) \simeq H^2(T_2, \mathbf{G}_\mathrm{m})$ contains non-trivial 2-torsion. Indeed, the cited theorem shows that this holds over $\overline{\mathbf{Q}}$, and applying it again shows that it holds for any algebraically closed $\mathbf{k}$ of characteristic zero, so that T is not retract rational.

In summary, we have the following result.

Theorem 5.1. *Suppose that* $\mathbf{k}$ *is algebraically closed of characteristic zero. Then the quartic threefold T admits a desingularisation*

$$f : \widetilde{T} \to T,$$

such that f is universally CH_0*-trivial. Moreover, we have* $\mathrm{Br}(\widetilde{T}) \neq 0$. □

Finally, we prove the relationship between the Brauer group and the ℓ-adic étale cohomology group which was used above.

Lemma 5.2. *Let X be a rationally connected* $\mathbf{k}$*-variety, with* $\operatorname{char} \mathbf{k} = 0$. *Then*

$$H^p(X, \mathcal{O}_X) = 0$$

for all $p > 0$.

Proof. See [Deb03, §3.4]. □

Let X be a variety over $\mathbf{C}$. The exact sequence of sheaves

$$0 \to \mathbf{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

on X (as an analytic space) induces a long exact sequence

$$\cdots \to H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \to H^2(X, \mathbf{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots.$$

The image of $\mathrm{Pic}(X)$ in $H^2(X, \mathbf{Z})$ is called the *Néron–Severi group*, and its rank, denoted by $\rho(X)$, is called the *Picard number* of X.

Lemma 5.3. *Let X be a rationally connected complex variety. Then the Picard number $\rho(X)$ is equal to the Betti number $b_2(X)$.* □

We fix some notations. For an abelian group A and a prime number ℓ, we denote

$$A\{\ell\} = \{ x \in A \mid \ell^n x = 0 \text{ for some } n \},$$

which is naturally a $\mathbf{Z}_\ell$-module. Suppose M is a $\mathbf{Z}_\ell$-module *of cofinite type*, i.e. one has

$$M \simeq (\mathbf{Q}_\ell/\mathbf{Z}_\ell)^{\oplus r} \oplus (\text{finite group}),$$

then we denote by M^{fin} its finite part, which is the largest finite submodule of M that is a direct summand.

Lemma 5.4. *Let X be a rationally connected complex variety, and let ℓ be a prime number. Then*

$$\text{Br}(X)\{\ell\} \simeq H^3(X, \mathbf{Z}_\ell(1))\{\ell\},$$

where the right hand side is the étale cohomology of the sheaf $\mathbf{Z}_\ell(1) = \varprojlim_n \boldsymbol{\mu}_{\ell^n}$, which may be identified with $\mathbf{Z}_\ell$ over $\mathbf{C}$.

Proof. By [Gro68, II, Theorem 3.1, p. 80], we have an exact sequence

$$0 \to \text{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell/\mathbf{Z}_\ell \longrightarrow H^2(X, \boldsymbol{\mu}_{\ell^\infty}) \longrightarrow \text{Br}(X)\{\ell\} \to 0.$$

Since the first term is a finite sum of copies of $\mathbf{Q}_\ell/\mathbf{Z}_\ell$, we have

$$\text{Br}(X)\{\ell\}^{\text{fin}} \simeq H^2(X, \boldsymbol{\mu}_{\ell^\infty})^{\text{fin}}.$$

By Lemma 5.3, the "corank" of $\text{Br}(X)\{\ell\}$ (i.e. number of summands $\mathbf{Q}_\ell/\mathbf{Z}_\ell$) is $b_2 - \rho = 0$, so that

$$\text{Br}(X)\{\ell\}^{\text{fin}} \simeq \text{Br}(X)\{\ell\}.$$

By [Gro68, III, (8.3), p. 144], we have an exact sequence

$$0 \to H^2(X, \boldsymbol{\mu}_{\ell^\infty})^{\text{fin}} \longrightarrow H^3(X, \mathbf{Z}_\ell(1)) \longrightarrow \varprojlim_n H^3(X, \boldsymbol{\mu}_{\ell^\infty})[\ell^n] \to 0,$$

where $[\ell^n]$ indicates the subgroup of elements killed by ℓ^n. Since the last term is torsion-free, we have

$$H^2(X, \boldsymbol{\mu}_{\ell^\infty})^{\text{fin}} \simeq H^3(X, \mathbf{Z}_\ell(1))\{\ell\},$$

whence the result follows. □

Consequences

Following Colliot-Thélène and Pirutka [CTP16], we present some consequences of the example given in the previous subsection.

The first result provides examples of smooth quartic threefolds over complex numbers, which are not retract rational.

Theorem 5.5. *Let $P \to \mathbf{P}^N_{\mathbf{C}}$ be the family of all quartic hypersurfaces in $\mathbf{P}^4_{\mathbf{C}}$. Let $t \in \mathbf{C} \setminus \overline{\mathbf{Q}}$ be a transcendental number. Then the set*

$$\left\{ z \in \mathbf{P}^N_{\mathbf{C}} \;\middle|\; \begin{array}{c} z \text{ has coordinates in } \overline{\mathbf{Q}}(t)\text{, and the hyper-} \\ \text{surface } P_z \text{ is smooth but not retract rational} \end{array} \right\}$$

is Zariski dense in $\mathbf{P}^N_{\mathbf{C}}$.

Proof. By Theorem 5.1, there is a quartic hypersurface $X \subset \mathbf{P}^4_{\overline{\mathbf{Q}}}$, with a desingularisation

$$f : \widetilde{X} \to X$$

such that f is universally CH_0-trivial, and $\mathrm{Br}(\widetilde{X}) \neq 0$.

Let $W \subset \mathbf{P}^N_{\overline{\mathbf{Q}}}$ be the closed subset corresponding to the singular hypersurfaces. Let $M \in \mathbf{P}^N_{\overline{\mathbf{Q}}}$ be the point corresponding to X. Choose a point $M' \in \mathbf{P}^N_{\overline{\mathbf{Q}}} \setminus W$, and let

$$L \simeq \mathbf{P}^1_{\overline{\mathbf{Q}}} \subset \mathbf{P}^N_{\overline{\mathbf{Q}}}$$

be the straight line connecting M and M', with generic point η. Then Theorem 4.9 implies that the quartic threefold X° defined by η, over the field $\overline{\mathbf{Q}}(x)$, is not geometrically retract rational. Moreover, by Theorem 4.11, for any embedding

$$\overline{\mathbf{Q}}(x) \hookrightarrow \mathbf{C},$$

the base change $X^\circ_{\mathbf{C}}$ is not retract rational, where a desingularisation of X° can be obtained via Hironaka's theorem, as is required by Theorem 4.11.

Let $R \in L(\mathbf{C})$ be a point whose coordinates are in $\overline{\mathbf{Q}}(t)$, but not in $\overline{\mathbf{Q}}$. Then the quartic threefold P_R is isomorphic to $X^\circ_{\mathbf{C}}$, for some embedding $\overline{\mathbf{Q}}(x) \hookrightarrow \mathbf{C}$. Indeed, we have a diagram of pull-back squares

$$\begin{array}{ccccccc}
P_R & \longrightarrow & P_{L_{\mathbf{C}}} & \longrightarrow & P_{\overline{\mathbf{Q}},\, L} & \longleftarrow & X^\circ \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} \mathbf{C} & \xrightarrow{R} & L_{\mathbf{C}} & \longrightarrow & L_{\overline{\mathbf{Q}}} & \xleftarrow{\eta} & \operatorname{Spec} \overline{\mathbf{Q}}(x)\,,
\end{array}$$

where $P_{\overline{\mathbf{Q}}} \to \mathbf{P}^N_{\overline{\mathbf{Q}}}$ denotes the family of all quartic hypersurfaces in $\mathbf{P}^4_{\overline{\mathbf{Q}}}$. By the choice of R, in the diagram, the image of $\operatorname{Spec} \mathbf{C}$ in $L_{\overline{\mathbf{Q}}}$ is the generic point (since any other point is a $\overline{\mathbf{Q}}$-rational point, and cannot be R). Therefore, the map $\operatorname{Spec} \mathbf{C} \to L_{\overline{\mathbf{Q}}}$ in the diagram factors through $\operatorname{Spec} \overline{\mathbf{Q}}(x)$, giving the desired embedding.

Therefore, P_R is not retract rational. This shows that every line passing through M and a point not in W contains infinitely many points where the hypersurface is not retract rational. This implies Zariski density. □

Together with results from §3, this result allows us to obtain general statements on the irrationality of quartic threefolds over complex numbers.

Theorem 5.6 (Colliot-Thélène and Pirutka)**.** *A very general quartic hypersurface in $\mathbf{P}^4_{\mathbf{C}}$ is not retract rational.*

Proof. By Theorem 3.12, and by Theorem 5.5. □

Theorem 5.7. *There are uncountably many stable equivalence classes in the family of quartic hypersurfaces in $\mathbf{P}^4_{\mathbf{C}}$.*

Proof. We apply Theorem 3.17.

We have seen that the family contains a threefold with no decomposition of the diagonal. But the (singular) hypersurface

$$X : x_0 x_1 (x_0 + x_1)(x_0 - x_1) = 0$$

has a decomposition of the diagonal. Indeed, let $z = (0 : 0 : 0 : 0 : 1) \in X$, and let $X_1, \dots, X_4$ be the irreducible components of X, each isomorphic to $\mathbf{P}^3$. The diagonal class of $X_i \times X_i$ is rationally equivalent to $[X_i] \times [z]$, up to a minor term D as before. Summing over i, we see that the diagonal of $X \times X$ is rationally equivalent to $[X] \times [z]$, up to a minor term. □

Remark 5.8. In these two theorems, the degree 4 can be replaced by any positive multiple of 4, since one can consider quartic threefolds in $\mathbf{P}^4$ with multiplicity $m > 1$, which will be a non-reduced hypersurface of degree $4m$. We use the fact that the Chow group of a "non-reduced variety" is isomorphic to that of its reduction [Ful98, Example 1.3.1].

Finally, we mention a result of Colliot-Thélène and Pirutka which provides examples of smooth quartic hypersurfaces in $\mathbf{P}^4_{\mathbf{C}}$, which are defined over $\overline{\mathbf{Q}}$, but not retract rational over $\mathbf{C}$.

Theorem 5.9. *There exist smooth quartic hypersurfaces in $\mathbf{P}^4_{\overline{\mathbf{Q}}}$ that are not universally* CH_0*-trivial over any field containing $\overline{\mathbf{Q}}$, and hence not retract rational over any field containing $\overline{\mathbf{Q}}$, and in particular, over* $\mathbf{C}$.

Proof. By Theorem 5.1, there is a singular quartic hypersurface $X \subset \mathbf{P}^4_{\overline{\mathbf{Q}}}$, with a desingularisation

$$f : \widetilde{X} \to X$$

such that f is universally CH_0-trivial, and $\mathrm{Br}(\widetilde{X})\,\{2\} \neq 0$. By Lemma 5.4, we thus have

$$H^3(\widetilde{X}, \mathbf{Z}_2)\,\{2\} \neq 0.$$

By Galois descent and by Lemma 4.8, we choose a finite extension $K/\mathbf{Q}$, over which X, $\widetilde{X}$ and f are defined, such that f is universally CH_0-trivial.

Let $\mathcal{O}_K$ be the ring of integers of K. Let $U \subset \operatorname{Spec} \mathcal{O}_K$ be an open set, such that there exists a map of U-schemes

$$\mathscr{f} : \widetilde{\mathcal{X}} \to \mathcal{X},$$

such that it coincides with $f : \widetilde{X} \to X$ over the generic point of U. Indeed, we define them to be cut out by the same set of equations as X and $\widetilde{X}$ in the projective space.

Shrinking U if necessary, we assume that U contains no 2-adic points, and that $\widetilde{\mathcal{X}}$ is smooth over U.

Shrinking U again, we assume for any closed point $v \in U$, the map of geometrical fibres

$$\mathscr{f}_{\overline{\kappa(v)}} : \widetilde{\mathcal{X}}_{\overline{\kappa(v)}} \to \mathcal{X}_{\overline{\kappa(v)}}$$

is a desingularisation map which is universally CH_0-trivial. In fact, it is what we get if we start with $\mathbf{k} = \overline{\kappa(v)}$ in §5.

Applying the smooth specialisation property of étale cohomology [Mil80, Corollary VI.4.2, p. 230], we see that for any $v \in U$,

$$H^3(\widetilde{\mathcal{X}}_{\overline{\kappa(v)}}, \mathbf{Z}_2) \simeq H^3(\widetilde{X}_{\overline{\mathbf{Q}}}, \mathbf{Z}_2).$$

It follows that $H^3(\widetilde{\mathcal{X}}_{\overline{\kappa(v)}}, \mathbf{Z}_2)\{2\} \neq 0$, and hence $\mathrm{Br}(\widetilde{\mathcal{X}}_{\overline{\kappa(v)}}) \neq 0$ by Lemma 5.4.

Fix a point $v \in U$. We regard v as a discrete valuation on K. By Lemma 4.6, there is an extension of discrete valuation rings

$$\mathcal{O}_{K,v} \subset A,$$

such that the residue field of A is $\overline{\kappa(v)}$. Let L be the fraction field of A.

Finally, there exists a smooth A-scheme whose special fibre is $\mathcal{X}_{\overline{\kappa(v)}}$. Indeed, in the projective space $\mathbf{P}_L^N$ parametrising the quartic hypersurfaces in $\mathbf{P}_L^4$, the set of points in $\mathbf{P}_L^N$ with coordinates in $\mathcal{O}_L$ lying over $\mathcal{X}_{\overline{\kappa(v)}}$ is Zariski dense. But there is an open set of $\mathbf{P}_L^N$ whose points correspond to smooth hypersurfaces.

Now we apply Theorem 4.11 to complete the proof. □

6 Example: Cubic threefolds

In this section, we consider a general example of a cubic hypersurface in $\mathbf{P}^4$, following [CTP16].

- Let $p \neq 3$ be a prime number.
- Consider one of the following two situations.

  - Let $\mathbf{k}$ be either a finite extension of $\mathbf{Q}_p$, or the field $\mathbf{F}_q((t))$, where q is a power of p. Let A be its ring of integers and $\mathbf{F}$ the finite residue field. Let $\pi \in A$ be a uniformising element.
  - Or, let $\mathbf{k}$ be a number field in which p is a prime. Let $A \subset \mathbf{k}$ be the corresponding discrete valuation ring, and let $\mathbf{F}$ be the finite residue field. Let $\pi = p$.

- Let $\mathbf{K}/\mathbf{k}$ be a cubic extension which is unramified at π, giving a cubic extension $\mathbf{E}/\mathbf{F}$ of residue fields.
- Let α be an element of the ring of integers of $\mathbf{K}$, such that $\mathbf{K} = \mathbf{k}(\alpha)$. Let $\beta \in \mathbf{E}$ be its image.
- Let

  $$\Phi \in A[u, v, w, x, y]$$

  be a cubic homogeneous polynomial, which defines a smooth hypersurface in $\mathbf{P}^4_A$.
- Let $\mathscr{X} \subset \mathbf{P}^4_A$ be the hypersurface cut out by the equation

  $$\Psi = \mathrm{Norm}_{\mathbf{K}/\mathbf{k}}(u + \alpha v + \alpha^2 w) + xy(x - y) + \pi^m \Phi(u, v, w, x, y) = 0,$$

  where $m > 0$ is an integer that is to be chosen.
- We can choose m so that the generic fibre

  $$X^\circ = \mathscr{X} \times_A \mathbf{k}$$

  is smooth over $\mathbf{k}$. In fact, the discriminant [Sal76, Article 105, p. 93] of Ψ is a polynomial in π^m, which is non-zero since the coefficient of its leading term is the discriminant of Φ. Therefore there are only finitely many values of m for which it is zero.
- We now look at the special fibre

  $$X = \mathscr{X} \times_A \mathbf{F},$$

  which is defined by the equation

  $$\mathrm{Norm}_{\mathbf{E}/\mathbf{F}}(u + \beta v + \beta^2 w) + xy(x - y) = 0.$$

  Let $\beta_1, \beta_2, \beta_3$ be the conjugates of β. Consider the linear coordinate change over $\mathbf{E}$

  $$(u, v, w) \mapsto (u + \beta_1 v + \beta_1^2 w,\ u + \beta_2 v + \beta_2^2 w,\ u + \beta_3 v + \beta_3^2 w). \qquad (6.0.1)$$

  The equation is now simplified over $\mathbf{E}$:

  $$uvw + xy(x - y) = 0.$$

Geometrically, this hypersurface has three singular points

$$(1:0:0:0:0),\quad (0:1:0:0:0),\quad (0:0:1:0:0). \tag{6.0.2}$$

They define a single point $M \in X$ with residue field $\mathbf{E}$.

- Let X_1 be the blow-up of X at M. Standard computation shows that under the map

  $$(X_1)_{\mathbf{E}} \to X_{\mathbf{E}},$$

  the inverse image of each of the 3 singular points is a union of two surfaces $\mathbf{P}^2_{\mathbf{E}}$, which intersect in a line $\mathbf{P}^1_{\mathbf{E}}$, which contains 3 singular points.

- The Galois group $G = \mathrm{Gal}(\mathbf{E}/\mathbf{F})$ acts on $X_{\mathbf{E}}$ by permuting u, v, w cyclically. It follows that $X_1 \simeq (X_1)_{\mathbf{E}}/G$, as a blow-up of X, has an exceptional divisor which is a union of two surfaces $\mathbf{P}^2_{\mathbf{E}}$, and contains 3 singular points with residue field $\mathbf{E}$.

- Let X_2 be the blow-up of X_1 at these three points. Standard computation shows that X_2 is smooth over $\mathbf{F}$, and that the exceptional divisor over each point is a rational surface.

- We conclude by Theorem 2.13 that the map

  $$X_2 \to X$$

  is a desingularisation map that is universally CH_0-trivial.

- We have $\mathrm{Br}(X_2) \neq 0$ by Theorem 6.4 below. Note that $\mathrm{Br}(\mathbf{F}) = 0$ by Wedderburn's theorem, so that $\mathrm{Br}(X_2)/\mathrm{Br}(\mathbf{F}) \neq 0$.

In summary, we have obtained the following theorem.

Theorem 6.1. *Let* $\mathbf{k}$ *be one of the following:*

- *a number field,*
- *a finite extension of* $\mathbf{Q}_p$ *with* $p \neq 3$*, or*
- *the field* $\mathbf{F}_q((t))$ *of characteristic not equal to* 3.

Then there exist smooth cubic hypersurfaces in $\mathbf{P}^4_{\mathbf{k}}$ *that is not universally* CH_0*-trivial over* $\mathbf{k}$*, and hence not retract rational over* $\mathbf{k}$.

Proof. By the above construction, and by Theorem 4.7, the generic fibre X°, as a smooth $\mathbf{k}$-variety, is not universally CH_0-trivial. □

Now, we complete the computation of the Brauer group of X_2.

Lemma 6.2. *Let* $\mathbf{K}$ *be a field, and let*

$$R = \mathbf{K}[u, v, w]/(uvw - 1).$$

Then an element of R *is invertible if and only if it is of the form* $t\,u^m v^n$*, with* $t \in \mathbf{K}$ *and* $m, n \in \mathbf{Z}$*.* □

Lemma 6.3. *Let* $p\colon X \to B$ *be a morphism of smooth varieties, where* B *is an integral curve. Suppose that*

- $\mathrm{Pic}(B) = 0$.
- *The Picard group of the generic fibre of* p *is zero.*
- *For each* $b \in B$ *such that* X_b *is not integral, every irreducible component of* X_b *is a principal divisor of* X*.*

Then $\mathrm{Pic}(X) = 0$.

Proof. Let $D \subset X$ be an irreducible divisor, with generic point η. We want to show that D is principal. There are two cases.

- $p(\eta)$ is a closed point $b \in B$. Then D is an irreducible component of the fibre X_b. If X_b is not irreducible, the result follows from the hypotheses. Otherwise, we have $D = X_b$, so that the rational function on B establishing the divisor $b \in B$ as principal, also establishes D as principal.
- $p(\eta)$ is the generic point. Then η defines a divisor of the generic fibre, which is principal by hypothesis. We thus obtain a rational function on X, whose divisor is the sum of D and some other divisors, each being an irreducible component of a fibre of p. We can thus apply the first case. □

Theorem 6.4. *Using the above notation, let* Y *be any desingularisation of* X*. For example, we may take* $Y = X_2$*. Then*

$$\mathrm{Br}(Y) \simeq \mathbf{Z}/3.$$

Proof. As before, we consider the coordinate change (6.0.1), so that $X_{\mathbf{E}}$ is defined by the equation

$$uvw = xy(x - y),$$

with the action of the Galois group $G = \mathrm{Gal}(\mathbf{E}/\mathbf{F}) \simeq \mathbf{Z}/3$ by permuting the coordinates u, v, w cyclically.

Let $U \subset X$ be the smooth locus, so that $U_{\mathbf{E}} \subset X_{\mathbf{E}}$ is the complement of the three singular points (6.0.2).

Let $V \subset U$ be the open set given by $xy \neq 0$. Then $U_{\mathbf{E}} \setminus V_{\mathbf{E}}$ consists of six irreducible components $\Delta_{u,x}$, $\Delta_{v,x}$, $\Delta_{w,x}$, $\Delta_{u,y}$, $\Delta_{v,y}$, $\Delta_{w,y}$, where for example, $\Delta_{u,x}$ is defined by $u = x = 0$. The group G acts on them by permuting u, v, w.

Let $p\colon V_{\mathbf{E}} \to \mathbf{A}^1_{\mathbf{E}} \setminus \{0\}$ be the projection given by $(u, v, w, x, y) \mapsto x/y$. The generic fibre is isomorphic to the surface $uvw = 1$ in the affine space $\mathbf{A}^3_{\mathbf{E}(x)}$, which is isomorphic to the open subset $uv \neq 0$ of $\mathbf{A}^2_{\mathbf{E}(x)}$, so that its Picard group is zero. Moreover, the only non-integral fibre is the fibre at 1, which consists of three irreducible components, each being a principal divisor of $V_{\mathbf{E}}$, since they are defined in $V_{\mathbf{E}}$ by the equations $u = 0$, $v = 0$ and $w = 0$ respectively. Applying Lemma 6.3, we obtain $\mathrm{Pic}(V_{\mathbf{E}}) = 0$.

Let $\mathrm{Div}_{U_{\mathbf{E}} \setminus V_{\mathbf{E}}}(U_{\mathbf{E}})$ denote the group of divisors of $U_{\mathbf{E}}$ supported in $U_{\mathbf{E}} \setminus V_{\mathbf{E}}$. It is a free abelian group of rank 6, generated by the divisors $\Delta_{u,x}$, etc. The canonical map

$$\beta\colon\ \mathrm{Div}_{U_{\mathbf{E}} \setminus V_{\mathbf{E}}}(U_{\mathbf{E}}) \to \mathrm{Pic}(U_{\mathbf{E}})$$

is surjective, as its image is the kernel of the restriction map $\mathrm{Pic}(U_{\mathbf{E}}) \to \mathrm{Pic}(V_{\mathbf{E}})$, the latter group being zero. The kernel of β consists of those divisors that are principal in $U_{\mathbf{E}}$. We thus have an exact sequence of G-modules

$$0 \to \mathbf{E}[V_{\mathbf{E}}]^\times/\mathbf{E}[U_{\mathbf{E}}]^\times \xrightarrow{\alpha} \mathrm{Div}_{U_{\mathbf{E}} \setminus V_{\mathbf{E}}}(U_{\mathbf{E}}) \xrightarrow{\beta} \mathrm{Pic}(U_{\mathbf{E}}) \to 0. \tag{6.4.1}$$

Let us take a closer look at the first term. Suppose $f \in \mathbf{E}[V_{\mathbf{E}}]^\times$. Using the projection $p\colon V_{\mathbf{E}} \to \mathbf{A}^1_{\mathbf{E}} \setminus \{0\}$ mentioned above, we can apply Lemma 6.2 to $\mathbf{K} = \mathbf{E}(x)$, to conclude that f has the form $f = t(x/y)\, u^m\, v^n$, for t a rational function, and $m, n \in \mathbf{Z}$. Since f has to be invertible on $V_{\mathbf{E}}$, we must have $m = n = 0$, and $t(x/y) = c\,(x/y)^k$ for some $c \in \mathbf{E}^\times$ and $k \in \mathbf{Z}$. It follows that $\mathbf{E}[V_{\mathbf{E}}]^\times \simeq \mathbf{E}^\times \oplus \mathbf{Z}$ and $\mathbf{E}[U_{\mathbf{E}}]^\times \simeq \mathbf{E}^\times$. The sequence (6.4.1) is thus

$$0 \to \mathbf{Z} \xrightarrow{\alpha} \mathbf{Z}[G] \oplus \mathbf{Z}[G] \xrightarrow{\beta} \mathrm{Pic}(U_{\mathbf{E}}) \to 0,$$

and the map α sends $k \in \mathbf{Z}$ to the divisor of the function $(x/y)^k$, which is the element $(k\varepsilon, -k\varepsilon) \in \mathbf{Z}[G] \oplus \mathbf{Z}[G]$, where $\varepsilon = \sum_{g\in G} g \in \mathbf{Z}[G]$.

Now is where the proof really begins. The exact sequence (6.4.1) will not be used anywhere in this proof; what we use is the sequence (6.4.1) with Y in place of U, where Y is a desingularisation of X as in the statement of this theorem. Such a sequence is obtained by a process as in the above argument. This gives an exact sequence of Tate cohomology groups

$$0 \to \widehat{H}^{-1}(G, \mathrm{Pic}(Y_{\mathbf{E}})) \xrightarrow{\delta} \widehat{H}^0(G, \overbrace{\mathbf{E}[V_{\mathbf{E}}]^\times/\mathbf{E}^\times}^{\simeq \mathbf{Z}}) \simeq \mathbf{Z}/3$$
$$\xrightarrow{\alpha'} \widehat{H}^0(G, \mathrm{Div}_{Y_{\mathbf{E}} \setminus V_{\mathbf{E}}}(Y_{\mathbf{E}})) \to \cdots,$$

where the $\widehat{H}^{-1}$ of $\mathrm{Div}_{Y_{\mathbf{E}} \setminus V_{\mathbf{E}}}(Y_{\mathbf{E}})$ vanishes, since the latter is a direct sum of copies of $\mathbf{Z}[G]$ and $\mathbf{Z}$, which both have zero $\widehat{H}^{-1}$ by direct computation.

The generator x/y of the second non-zero term goes to a divisor of $Y_{\mathbf{E}}$ which is the norm (in the G-module sense) of a divisor of $Y_{\mathbf{E}}$. Indeed, we have seen that the divisor of the rational function x/y on $U_{\mathbf{E}}$ is the norm of an element. But $Y_{\mathbf{E}} \setminus U_{\mathbf{E}}$ is the inverse image of the three singular points of $X_{\mathbf{E}}$, and the action of G permutes these three parts of $Y_{\mathbf{E}} \setminus U_{\mathbf{E}}$. Therefore, the divisor of x/y on $Y_{\mathbf{E}} \setminus U_{\mathbf{E}}$ is the norm of the divisor of x/y on one of these three parts. This shows that $\alpha'(x/y) = 0$, so that x/y is in the image of δ. It follows that $\widehat{H}^{-1}(G, \mathrm{Pic}(Y_{\mathbf{E}})) \simeq \mathbf{Z}/3$.

By [CTS77, Lemma 15], there is a short exact sequence

$$0 \to \mathrm{Br}(\mathbf{F}, \mathbf{E}) \longrightarrow \mathrm{Br}(Y, \mathbf{E}) \longrightarrow H^1(G, \mathrm{Pic}(Y_{\mathbf{E}})) \to 0,$$

where $\mathrm{Br}(Y, \mathbf{E}) = \ker(\mathrm{Br}(Y) \to \mathrm{Br}(Y_{\mathbf{E}}))$, and similarly for $\mathrm{Br}(\mathbf{F}, \mathbf{E})$. Since $\mathbf{F}$ and $\mathbf{E}$ are finite, one has $\mathrm{Br}(\mathbf{F}, \mathbf{E}) = 0$ by Wedderburn's theorem. Since G is cyclic, one has $H^1(G, \mathrm{Pic}(Y_{\mathbf{E}})) \simeq \widehat{H}^{-1}(G, \mathrm{Pic}(Y_{\mathbf{E}})) \simeq \mathbf{Z}/3$. It follows that the middle term is $\mathbf{Z}/3$. In other words, there is a short exact sequence

$$0 \to \mathbf{Z}/3 \longrightarrow \mathrm{Br}(Y) \longrightarrow \mathrm{Br}(Y_{\mathbf{E}}) \to 0.$$

Recall the projection $p\colon V_{\mathbf{E}} \to \mathbf{A}^1_{\mathbf{E}}$ defined above. We have seen that the inverse image of the complement of $\{0, 1\}$ is isomorphic to the subset of $\mathbf{A}^3_{\mathbf{E}}$ defined by $uvx(x-1) \neq 0$. This shows that $Y_{\mathbf{E}}$ is rational, so that $\mathrm{Br}(Y_{\mathbf{E}}) \simeq \mathrm{Br}(\mathbf{E}) \simeq 0$ by Theorem 2.23 and Wedderburn's theorem. Therefore, $\mathrm{Br}(Y) \simeq \mathbf{Z}/3$. □

7 Example: Quadric surface bundles

This section presents the result of Hassett, Pirutka and Tschinkel [HPT16], which states that over complex numbers, a very general fourfold which is a quadric surface bundle over $\mathbf{P}^2$ is not retract rational, while those fourfolds that are rational are *dense* in the family, in euclidean topology.

Definition 7.1. Let S be an integral surface. A *quadric surface bundle* over S, is a fourfold $X \subset S \times \mathbf{P}^3$, such that the composition

$$\pi\colon X \hookrightarrow S \times \mathbf{P}^3 \xrightarrow{\mathrm{pr}_1} S$$

is flat with smooth generic fibre.

Irrationality

We are interested in the case $S = \mathbf{P}^2$, and we consider the family of all hypersurfaces of $\mathbf{P}^2 \times \mathbf{P}^3$ of bidegree $(2, 2)$. A general member of this family is a quadric surface bundle over $\mathbf{P}^2$.

As before, we use a specific example to establish the irrationality of a very general member.

- Consider the fourfold $X \subset \mathbf{P}^2 \times \mathbf{P}^3$, given by

$$yzs^2 + xzt^2 + xyu^2 + F(x, y, z)v^2 = 0,$$

  where x, y, z are the coordinates of $\mathbf{P}^2$, and s, t, u, v the coordinates of $\mathbf{P}^3$, with

$$F(x, y, z) = x^2 + y^2 + z^2 - 2yz - 2xz - 2xy.$$

- Hassett, Pirutka and Tschinkel [HPT16, §5] constructed a universally CH_0-trivial desingularisation of X.

Let A be a discrete valuation ring, with valuation v, fraction field $\mathbf{K}$, and residue field κ. There is a residue map

$$\partial_v : H^2(\mathbf{K}, \mathbf{Z}/2) \to H^1(\kappa, \mathbf{Z}/2) \simeq \kappa^\times/\kappa^{\times 2},$$

which sends

$$(a, b) \mapsto (-1)^{v(a)\,v(b)} a^{v(b)}/b^{v(a)},$$

where $a, b \in \mathbf{K}^\times$, and $(a, b) = a \cup b$ is the cup product of a and b. The kernel of ∂_v coincides with the image of $H^2(\operatorname{Spec} A, \mathbf{Z}/2)$, so that an element of $H^2(\mathbf{K}, \mathbf{Z}/2)$ is unramified (Definition 2.20) if and only if it is in the kernel of ∂_v for all v. See [CT95] for more details.

Proposition 7.2. $\operatorname{Br}(X)$ *contains non-trivial* 2*-torsion. In fact, let*

$$\alpha = (x/z, y/z) \in \operatorname{Br}(\mathbf{C}(\mathbf{P}^2))\,[2],$$

and let $\alpha' \in \operatorname{Br}(\mathbf{C}(X))$ *be its image. Then* α' *is non-zero and unramified, i.e., lies in* $\operatorname{Br}(X)$.

Proof. The generic fibre X° of $X \to \mathbf{P}^2$ is a quadric surface over the field $\mathbf{K} = \mathbf{C}(x/z, y/z)$, and its discriminant is not a square in $\mathbf{K}$. Applying [CTS19, Proposition 6.2.3 (c)], the natural map

$$i : \operatorname{Br}(\mathbf{K}) \to \operatorname{Br}(X^\circ)$$

is an isomorphism. As $\mathbf{K}(X^\circ) \simeq \mathbf{C}(X)$, it remains to show that α' is unramified, i.e., $\partial_v(\alpha') = 0$ for all valuations v on $\mathbf{C}(X)/\mathbf{C}$.

Let us first look at the residues of α. By definition, only the following residues are non-trivial:

- $\partial_x(\alpha) = y/z \in \mathbf{C}(y/z)^\times/\mathbf{C}(y/z)^{\times 2}$, along the line $L_x : x = 0$.
- $\partial_y(\alpha) = x/z \in \mathbf{C}(x/z)^\times/\mathbf{C}(x/z)^{\times 2}$, along the line $L_y : y = 0$.
- $\partial_z(\alpha) = x/y \in \mathbf{C}(x/y)^\times/\mathbf{C}(x/y)^{\times 2}$, along the line $L_z : z = 0$.

Now let v be a valuation on $\mathbf{C}(X)/\mathbf{C}$. We need to show that $\partial_v(\alpha') = 0$.

Let $\mathcal{O}_v \subset \mathbf{C}(X)$ be the valuation ring of v. If $\mathcal{O}_v$ contains $\mathbf{K}$, then $\partial_v(\alpha') = 0$. Therefore, if we consider the centre of v in $\mathbf{P}^2$, there are two remaining cases.

- *The centre is the generic point of a curve* $C \subset \mathbf{P}^2$. The inclusion of discrete valuation rings $\mathcal{O}_{\mathbf{P}^2, C} \subset \mathcal{O}_v$ induces a commutative diagram

$$\begin{array}{ccc} \alpha \in H^2(\mathbf{K}, \mathbf{Z}/2) & \xrightarrow{\partial_v} & H^1(\kappa(C), \mathbf{Z}/2) \\ \downarrow & & \downarrow \\ \alpha' \in H^2(\mathbf{C}(X), \mathbf{Z}/2) & \xrightarrow{\partial_v} & H^1(\kappa(v), \mathbf{Z}/2)\,. \end{array}$$

It follows that if C is different from L_x, L_y or L_z, then $\partial_v(\alpha') = 0$, since $\partial_v(\alpha) = 0$. If, for example, $C = L_x$, then $\partial_v(\alpha') = y$ in the residue field

$$\mathbf{C}(y,t,u)\big[s = \sqrt{F(0,y,1)/y}\big] \simeq \mathbf{C}\big(\sqrt{y},t,u\big),$$

where we have set $z = 1$ and $v = 1$. Therefore, $\partial_v(\alpha')$ is a square in the residue field, and hence is trivial. (The key point is that $F(x,y,z)$ is a square modulo any one of x, y, z.)

- *The centre is a closed point* $P \in \mathbf{P}^2$. There are three cases.
  - (i) $P \notin L_x \cup L_y \cup L_z$. Then $v(x/z) = v(y/z) = 0$, so that $\partial_v(\alpha') = 0$.
  - (ii) P lies on one of the three lines, say L_x. Then $y/z \neq 0$ at P, so that y/z is a square in the completion $\widehat{\mathcal{O}_{\mathbf{P}^2,P}}$, which embeds in $\widehat{\mathcal{O}_v}$, whose fraction field is the completion $\mathbf{C}(X)_v$. Thus y/z is a square in $\mathbf{C}(X)_v$, and $\alpha' = 0$ in $H^2(\mathbf{C}(X)_v, \mathbf{Z}/2)$, so that $\partial_v(\alpha') = 0$.
  - (iii) P lies on two of the three lines, say L_x and L_y. As in the previous case, $F(x,y,z)/z^2$ is a square in the completion $\widehat{\mathbf{K}}$. Applying [CTS19, Proposition 6.2.3 (c)] to the quadric $X^\circ_{\widehat{\mathbf{K}}}$, we see that the image of α in $H^2(\widehat{\mathbf{K}}(X^\circ), \mathbf{Z}/2)$ is zero. The natural map of fields $\widehat{\mathbf{K}}(X^\circ) \to \mathbf{C}(X)_v$ shows that α is zero in $H^2(\mathbf{C}(X)_v, \mathbf{Z}/2)$. Therefore, $\partial_v(\alpha') = 0$. □

Applying Theorem 2.23, and applying Theorem 3.12 to the family of bidegree $(2,2)$ hypersurfaces in $\mathbf{P}^2 \times \mathbf{P}^3$, we obtain the following.

Corollary 7.3. *A very general bidegree* $(2,2)$ *hypersurface in* $\mathbf{P}^2 \times \mathbf{P}^3$ *is not retract rational.* □

Density of the rational locus

Now, we begin to prove a remarkable fact about this example, that those rational members in the family of quadric surface bundles over $\mathbf{P}^2$ is dense. This also shows that in Theorem 3.12, "a countable union of closed sets" can not be improved to "a closed set".

By a *multisection* of X/S degree d, we mean a family of 0-cycles of degree d, in the sense of Definition 3.1.

Lemma 7.4. *Let S be a rational surface, and $X \to S$ a quadric surface bundle. Suppose that X/S has a multisection of odd degree. Then X is rational.*

Proof. X is rational, if and only if the generic fibre X° is rational over the field $\mathbf{C}(S)$. Since X° is a smooth quadric surface, it is rational if and only if it has a rational point, as the projection from a rational point will give a birational map between X° and $\mathbf{P}^2$. Thus it suffices to show that X° has a $\mathbf{C}(S)$-rational point.

By a theorem of Springer [Spr52], X° has a $\mathbf{C}(S)$-rational point, if and only if X° has a $\mathbf{K}$-rational point for some extension $\mathbf{K}/\mathbf{C}(S)$ of odd degree. Thus, we only need to show that X° has a 0-cycle of odd degree, which will imply that X° has a closed point of odd degree.

But by hypothesis, X/S has a multisection of odd degree, which gives rise to a 0-cycle of X° of odd degree. □

For a quadric surface bundle $X \to S$, and an *integral* $(2,2)$-*class*, that is, an element $\alpha \in H^{2,2}(X) \cap H^4(X;\mathbf{Z})$, we say that α *meets the fibre* X_s *in degree* d, where $s \in S$, if the pairing of α with the homology class of X_s equals d.

Lemma 7.5. *Let S be a rational surface, and $\pi : X \to S$ a quadric surface bundle. Suppose that X has an integral* $(2,2)$-*class meeting the fibres of π in odd degree. Then X is rational.*

Proof. Let $S_0 \subset S$ be the locus where the rank of the quadratic form is ≥ 3 (the full rank is 4), and let $X_0 = X \times_S S_0$.

Let $F_1 \to S$ be the relative variety of lines of π, i.e., the points of the fibre $(F_1)_s$ correspond to straight lines contained in the fibre X_s. When X_s is non-degenerate, it contains 2 families of lines, each parametrised by $\mathbf{P}^1$. When the rank of the quadratic form drops by 1, X_s becomes a quadric cone, which contains 1 family of lines parametrised by $\mathbf{P}^1$.

This shows that $F_1|_{S_0} \to S_0$ factors as

$$F_1|_{S_0} \xrightarrow{p} T_0 \longrightarrow S_0,$$

where p is an étale $\mathbf{P}^1$-bundle, and $T_0 \to S_0$ is a double cover branched along $S_0 \cap D$, where $D \subset \mathbf{P}^2$ is the locus of degenerate fibres.

Let F be a desingularisation of the closure of $F_1|_{S_0}$ in F_1. The correspondence $\Gamma_1 = \{(x,\ell) \mid x \in \ell\} \subset X \times_S F_1$ induces a correspondence Γ from X to F, which induces a map

$$\Gamma_* : H^{2,2}(X) \to H^{1,1}(F).$$

On the other hand, let η be the generic point of S. There is a map

$$\Xi_* : \operatorname{Pic}(F_\eta) \simeq \mathrm{CH}_0(F_\eta) \to \mathrm{CH}_0(X_\eta)$$

constructed as follows. For a divisor $Z \subset F_\eta$, i.e. a choice of n lines from each family of lines on each quadric surface, let $\Xi_*(Z) \subset X_\eta$ be the n^2 points where these lines intersect. Note that Ξ_* sends a divisor of odd degree on each geometric component of F_η to a multisection of odd degree.

Now let us prove the lemma. By hypothesis, X has an integral $(2,2)$-class meeting the fibres in odd degree. Applying the map Γ_*, we obtain an integral $(1,1)$-class of F of meeting the fibres in odd degree. By the Lefschetz theorem on $(1,1)$-classes [GH94, p. 163], F has a divisor which meets the fibres in odd degree. Finally, applying the map Ξ_* to this divisor, we obtain a multisection of X/S which meets the fibres in odd degree. Applying Lemma 7.4 completes the proof. □

Next, we analyse the Hodge classes in the case $S = \mathbf{P}^2$, in order to verify the assumption of this lemma. The key tool is the following technique of Voisin.

Lemma 7.6. *Let $Y \to B$ be a flat, projective family of complex varieties. Suppose there exists $b \in B$ and $\lambda \in H^{p,p}(Y_b, \mathbf{R})$, such that the infinitesimal period map*

$$\overline{\nabla}(\lambda) : T_{B,b} \to H^{p-1,p+1}(Y_b)$$

is surjective, where $T_{B,b}$ denotes the tangent space of B at b. Then for any open set $U \subset B$ (in euclidean topology) *containing b, such that $Y|_U \to U$ is a trivial bundle, the map* (notations are explained below)

$$\phi : \mathscr{H}^{p,p}_{\mathbf{R}}|_U \hookrightarrow \mathscr{H}^{2p}_{\mathbf{R}}|_U \simeq H^{2p}(Y_b, \mathbf{R}) \times U \to H^{2p}(Y_b, \mathbf{R}) \to F^{p-1}H^{2p}(Y_b, \mathbf{R})$$

is submersive at λ.

We use the notation $\mathscr{H}^{p,q}$, $\mathscr{H}^{p,q}_{\mathbf{R}}$, etc., to refer to the vector bundles over B, whose fibres are the cohomology of the fibres of $Y \to B$. The notation $F^{p-1}H^{2p}$ refers to the Hodge filtration, and in this case, it is equal to $H^{p-1,p+1} \oplus H^{p,p} \oplus \cdots \oplus H^{2p,0}$.

Proof. See, for example, [Voi07, §5.3.4]. □

In the following, we use the notation

$$Y \to B$$

for the family of all smooth bidegree (2, 2) hypersurfaces in $\mathbf{P}^2 \times \mathbf{P}^3$, and the notation Y_b refers to its fibres.

Proposition 7.7. *The Hodge and Betti numbers of Y_b are given by*

- $b_0 = b_8 = 1$.
- $b_1 = b_3 = b_5 = b_7 = 0$.
- $b_2 = h_{1,1} = b_6 = h_{3,3} = 2$.
- $b_4 = 46$, $h_{0,4} = h_{4,0} = 0$, $h_{1,3} = h_{3,1} = 3$, $h_{2,2} = 40$.

Proof. The Lefschetz hyperplane theorem shows that

$$b_k(Y_b) = b_k(\mathbf{P}^2 \times \mathbf{P}^3) \quad \text{and} \quad h_{p,q}(Y_b) = h_{p,q}(\mathbf{P}^2 \times \mathbf{P}^3)$$

for $k < 4$ and $p + q < 4$. This, together with Poincaré/Serre duality, gives the first three items.

We compute b_4 by analysing the map $Y_b \to \mathbf{P}^2$. Let $D \subset \mathbf{P}^2$ be the locus of degenerate fibres. If Y_b is defined by the equation

$$\textstyle\sum_{i,j=0}^{2} \sum_{k,l=0}^{3} a_{ijkl} x_i x_j y_k y_l = 0,$$

where the coefficients a_{ijkl} are assumed to be symmetric with respect to i, j and k, l, then D is cut out by the equation

$$\det\left(\sum_{i,j=0}^{2} a_{ijkl}x_i x_j\right)_{\substack{0\le k\le 3\\ 0\le l\le 3}} = 0.$$

Therefore, D is an octic curve, and hence has genus 21 and Euler number -40.

Recall that for a complex variety X and a closed subvariety $Z \subset X$, we have an additive formula $\chi(X) = \chi(Z) + \chi(X \setminus Z)$ of Euler numbers. Hence we have

$$\begin{aligned}\chi(Y_b) &= \chi(\mathbf{P}^1 \times \mathbf{P}^1)\,\chi(\mathbf{P}^2 \setminus D) + \chi(\text{quadric cone})\,\chi(D)\\ &= 4\cdot(3-(-40)) + 3\cdot(-40) = 52,\end{aligned}$$

and it follows that $b_4(Y_b) = \chi(Y_b) - b_0 - b_2 - b_6 - b_8 = 46$.

To compute the remaining Hodge numbers, we apply the result of Batyrev and Cox on hypersurfaces in toric varieties [BC94, Theorem 10.13], which implies that the *vanishing cohomology*, defined by

$$H^{p,q}(Y_b)_{\mathrm{van}} = H^{p,q}(Y_b)/H^{p,q}(\mathbf{P}^2 \times \mathbf{P}^3)$$

is given by the formula

$$H^{p,4-p}(Y_b)_{\mathrm{van}} \simeq \mathrm{Jac}(F)_{(7-2p,6-2p)},$$

where F is the defining equation of Y_b, and

$$\mathrm{Jac}(F) = \mathbf{C}[x,y,z;s,t,u,v]/\mathcal{J}(F)$$

is the $\mathbf{Z}^2$-graded Jacobian ring of F, where $\mathcal{J}(F)$ is the ideal generated by the partial derivatives of F.

Using this method, we obtain

- $h_{4,0} = \dim \mathrm{Jac}(F)_{(-1,-2)} = 0$, and hence $h_{0,4} = 0$ as well.
- $h_{3,1} = \dim \mathrm{Jac}(F)_{(1,0)} = 3$, and hence $h_{1,3} = 3$ as well.
- $h_{2,2} = b_4 - (h_{0,4} + h_{1,3} + h_{3,1} + h_{4,0}) = 40$. □

Corollary 7.8. *There exists $b \in B$ which satisfies the assumption of Lemma* 7.6*, with $p = 2$.*

Proof. Since $B \subset \mathbf{P}(\mathbf{C}[x,y,z;s,t,u,v]_{(2,2)})$, we may identify

$$T_{B,b} \simeq \mathbf{C}[x,y,z;s,t,u,v]_{(2,2)}/(\mathbf{C}\cdot F),$$

where F is the defining equation of Y_b. The infinitesimal period map

$$\overline{\nabla}: T_{B,b} \times H^{2,2}(Y_b) \to H^{1,3}(Y_b)$$

is given by multiplication

$$(\mathbf{C}[x,y,z;s,t,u,v]/(F))_{(2,2)} \times \mathrm{Jac}(F)_{(3,2)} \to \mathrm{Jac}(F)_{(5,4)},$$

by [Voi07, Theorem 6.13], which applies by the identifications [BC94, Corollary 10.2, Theorem 10.6, and Theorem 10.13]. We consider the fibre Y_b given by

$$F = x^2s^2 + y^2t^2 + z^2u^2 + yzs^2 + xzt^2 + xyu^2 + x^2sv + y^2tv + z^2uv = 0.$$

One verifies that it is a smooth hypersurface in $\mathbf{P}^2 \times \mathbf{P}^3$, and that $\mathrm{Jac}(F)_{(5,4)}$ is generated by the basis elements xz^4v^4, yz^4v^4 and z^5v^4, using computer software. Therefore, if we take $\lambda = z^3v^2 \in \mathrm{Jac}(F)_{(3,2)}$, then the map

$$\cdot\,\lambda\colon\ \mathbf{C}[x,y,z;s,t,u,v]_{(2,2)} \to \mathrm{Jac}(F)_{(5,4)}$$

is surjective. □

Theorem 7.9. *The set of those $b \in B$ such that Y_b has an integral $(2,2)$-class meeting the fibres of $Y_b \to \mathbf{P}^2$ in odd degree is dense in B, in euclidean topology.*

Proof. Instead of finding an integral class, we only need to find such a class with the coefficient ring

$$R = \{m/n \mid m,n \in \mathbf{Z},\ 2 \nmid n\},$$

as we can multiply by an odd integer to turn such a class into an integral class. (We have an obvious definition of an odd element in R.)

Let $b_0 \in B$ be as in the previous corollary, and let $U \subset B$ be an open set which trivialises $Y \to B$ near b_0. Such a trivialisation preserves the homology classes of the fibres of $Y_b \to \mathbf{P}^2$.

We have shown that $H^{0,4}(Y_{b_0}) = 0$. Thus, Lemma 7.6 shows that the image of the map

$$\phi\colon\ \mathscr{H}^{2,2}_{\mathbf{R}}|_U \to H^4(Y_{b_0}, \mathbf{R})$$

contains an open set. Since the image consists of those classes that are of type $(2,2)$ over some $b \in U$, it suffices to show that the elements of $H^4(Y_{b_0}, R)$ that meet the fibres of $Y_{b_0} \to \mathbf{P}^2$ in odd degree are dense in $H^4(Y_{b_0}, \mathbf{R})$, so that one such element lies in the image. We only need to prove this for $b_0 \in B$, as the set of such b_0 is Zariski open in B.

The quadric surface bundle Y_{b_0} has a constant section $\mathbf{P}^2 \to Y_{b_0}$ given by $s = t = u = 0$. This gives rise to an element $\alpha \in H^4(Y_{b_0}, \mathbf{Z})$, which intersects the fibres of $Y_{b_0} \to \mathbf{P}^2$ in degree 1. For any $\beta \in H^4(Y_{b_0}, R)$, the class $\alpha + 2\beta$ also intersects the fibres of $Y_{b_0} \to \mathbf{P}^2$ in odd degree. Such classes are dense in $H^4(Y_{b_0}, R)$. □

The results are summarised as follows.

Theorem 7.10. *In the family of all bidegree $(2,2)$ hypersurfaces in $\mathbf{P}^2 \times \mathbf{P}^3$, a very general member is not retract rational, while the rational members form a dense subset in the family, in euclidean topology.* □

References

[AM72] M. Artin and D. Mumford (1972). “Some elementary examples of unirational varieties which are not rational”. *Proceedings of the London Mathematical Society* **25** (3), 75–95.

[BC94] V. V. Batyrev and D. A. Cox (1994). “On the Hodge structure of projective hypersurfaces in toric varieties”. *Duke Mathematical Journal* **75** (2), 293–338.

[Blo10] S. Bloch (2010). *Lectures on Algebraic Cycles*. 2nd ed. New Mathematical Monographs **16**. Cambridge University Press.

[Bou06] N. Bourbaki (2006). *Algèbre Commutative, Chapitres 8 et 9*. Springer.

[CT05] J.-L. Colliot-Thélène (2005). “Un théorème de finitude pour le groupe de Chow des zéro-cycles d'un groupe algébrique linéaire sur un corps p-adique”. *Inventiones mathematicae* **159**, 589–606.

[CT95] J.-L. Colliot-Thélène (1995). “Birational invariants, purity and the Gersten conjecture”. *K-Theory and Algebraic Geometry: Connections with Quadratic Forms and Division Algebras, Part 1*. Proceedings of Symposia in Pure Mathematics **58**. AMS, 1–64.

[CTP16] J.-L. Colliot-Thélène and A. Pirutka (2016). “Hypersurfaces quartiques de dimension 3 : non-rationalité stable”. *Annales Scientifiques de l'École Normale Supérieure* **49** (2), 371–397.

[CTS19] J.-L. Colliot-Thélène and A. N. Skorobogatov (2019). *The Brauer–Grothendieck group*. Preprint.

[CTS77] J.-L. Colliot-Thélène and J.-J. Sansuc (1977). “La R-équivalence sur les tores”. *Annales Scientifiques de l'École Normale Supérieure* **10** (2), 175–229.

[Deb03] O. Debarre (2003). “Variétés rationnellement connexes”. *Astérisque* **290**, 243–266.

[EGA-II] A. Grothendieck (1961). “Éléments de géométrie algébrique II. Étude globale élémentaire de quelques classes de morphismes”. *Publications mathématiques de l'I.H.É.S.* **8**, 5–222.

[EGA-IV3] A. Grothendieck (1966). “Éléments de géométrie algébrique IV. Étude locale des schémas et des morphismes de schémas. 3e partie”. *Publications mathématiques de l'I.H.É.S.* **28**, 5–255.

[EKW16] H. Esnault, M. Kerz, and O. Wittenberg (2016). “A restriction isomorphism for cycles of relative dimension zero”. *Cambridge Journal of Mathematics* **4** (2), 163–196.

[FAG] B. Fantechi et al. (2005). *Fundamental Algebraic Geometry: Grothendieck's FGA Explained*. Mathematical Surveys and Monographs **123**. AMS.

[Ful98] W. Fulton (1998). *Intersection Theory*. 2nd ed. Ergebnisse der Mathematik und ihrer Grenzgebiete **2**. Springer.

[GH94] P. Griffiths and J. Harris (1994). *Principles of Algebraic Geometry*. Wiley Classics Library. John Wiley & Sons.

[Gro68] A. Grothendieck (1968). “Le groupe de Brauer I, II, III”. *Dix exposés sur la cohomologie des schémas*. Masson & North-Holland, 46–188.

[HPT16] B. Hassett, A. Pirutka, and Yu. Tschinkel. *Stable rationality of quadric surface bundles over surfaces*. arXiv: `1603.09262`.

[Kol96] J. Kollár (1996). *Rational curves on algebraic varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete **32**. Springer.

[Mil80] J. S. Milne (1980). *Étale Cohomology*. Princeton Mathematical Series **33**. Princeton University Press.

[Ros96] M. Rost (1996). “Chow groups with coefficients”. *Documenta Mathematica* **1**, 319–393.

[Sal76] G. Salmon (1876). *Lessons Introductory to the Modern Higher Algebra*. 3rd ed. Dublin.

[Ser79] J.-P. Serre (1979). *Local Fields*. Trans. by M. J. Greenberg. Graduate Texts in Mathematics **67**. Springer.

[Spr52] T. A. Springer (1952). “Sur les formes quadratiques d’indice zéro”. *C. R. Acad. Sci. Paris* **234**, 1517–1519.

[Voi07] C. Voisin (2007). *Hodge Theory and Complex Algebraic Geometry II*. Trans. by L. Schneps. Cambridge Studies in Advanced Mathematics **77**. Cambridge University Press.

[Voi15] C. Voisin (2015). “Unirational threefolds with no universal codimension 2 cycle”. *Inventiones mathematicae* **201** (1), 207–237.