A Generalised Handle Theory

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ABSTRACT

We introduce a variant of the notion of a handlebody, in order to apply handle theory to non-compact manifolds. As an application, we classify all 2-manifolds with finite topology, obtaining a new proof that $\mathbb{R}^2$ has a unique smooth structure.

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Convention. In this paper, the word “manifold” refers to a smooth manifold, possibly with boundary, unless otherwise mentioned. The boundary of a submanifold is not assumed to be contained in the boundary of the ambient manifold.

1 Introduction

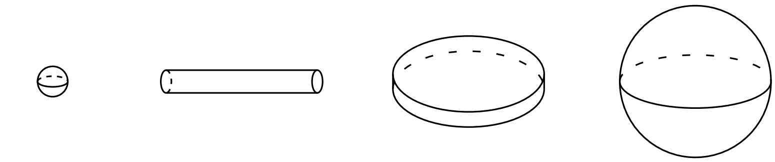
Handle theory is a powerful tool in differential topology. Its main idea is to use *handles* as basic building blocks for manifolds, just like cells are building blocks for CW complexes.

Definition 1.1. Let $0 \leq \lambda \leq n$ be integers. A *λ-handle* of dimension n is a thickened version of the λ-disk:

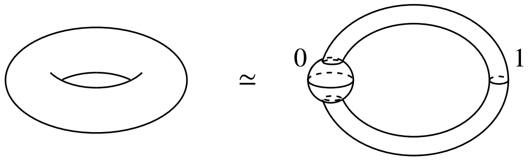
$$h^\lambda := D^\lambda \times D^{n-\lambda}.$$

It is an n-dimensional manifold, possibly with corners. ◁

For example, for $n = 3$, the different types of handles are depicted below.



Every manifold can be obtained from 0-handles by attaching other handles. For example, the solid torus $D^2 \times S^1$ decomposes into a 0-handle and a 1-handle.



A formal definition goes as follows.

Definition 1.2. A *relative n-handlebody* is a sequence

$$A = N_0 \subset N_1 \subset \cdots$$

of smooth n-manifolds, possibly with boundary, such that each N_i is obtained from N_{i-1} by attaching a λ-handle:

$$N_i \simeq N_{i-1} \cup_{\Phi_i} h^\lambda,$$

where λ may vary with i, and the attaching map

$$\Phi_i : \partial D^\lambda \times D^{n-\lambda} \to \partial N_{i-1}$$

is a smooth embedding. We require local finiteness (see below), so that the space

$$N := \bigcup_i N_i$$

is a smooth n-manifold.

By abuse of language, the pair (N, A) is called a *relative n-handlebody*. If $A = \varnothing$, then N is called an *n-handlebody*. It is *finite* if the above sequence is finite. ◁

In this definition, *local finiteness* means that every point in N has a neighbourhood that intersects with only finitely many handles. This is needed to ensure that N is a manifold.

This definition is not completely rigorous, since attaching a handle will form corners on the manifold, and one needs to eliminate them in each step. For details, the reader is referred to [Wal16].

The handlebody was invented by Smale, and played a magical role in his proof of the generalised Poincaré conjecture [Sma61].

Theorem 1.3 (Generalised Poincaré conjecture)**.** *If $n \geq 6$, then every smooth n-manifold homotopy equivalent to the n-sphere is homeomorphic to the n-sphere.*

Note that "homeomorphic" can not be improved to "diffeomorphic", since there exist different smooth structures on S^7.

The main step of the proof was the following.

Theorem 1.4 (h-cobordism theorem)**.** *Suppose M, N, W are simply connected compact manifolds, with* $\dim W \geq 6$ *and*

$$\partial W = M \sqcup N.$$

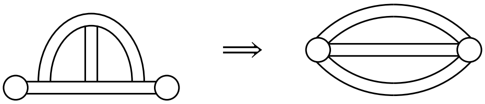
If the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences, then there is a diffeomorphism

$$W \simeq M \times [0, 1].$$

Sketch of Proof. The proof of this theorem relies on the theory of handlebodies. We first decompose W into handles, regarding it as obtained from M by attaching handles. For dimensional reasons, M is replaced by $M \times [0, 1]$, which is a neighbourhood of M in W. Then we manipulate these handles using the following two theorems. It turns out that everything can be cancelled out perfectly, leading to the desired diffeomorphism. □

The two theorems that were used in the proof of the h-cobordism theorem are the following.

Theorem 1.5 (Rearrangement)**.** *Every finite handlebody can be rearranged, so that every λ-handle is attached on handles of type $< \lambda$.*



To be more precise, it is convenient to introduce some terminology here. (They are invented by the author and not meant to be used elsewhere.)

Definition 1.6. Two n-handlebodies are *similar* if their manifolds N are diffeomorphic, and for every λ, they have the same number of λ-handles. ◁

For example, the two handlebodies shown in the above picture are similar.

Definition 1.7. A handlebody is *good* if every λ-handle is attached on handles of type $< \lambda$. ◁

Thus, the rearrangement theorem can be reformulated as follows.

Theorem 1.5′ (Rearrangement). *Every finite handlebody is similar to a good one.*

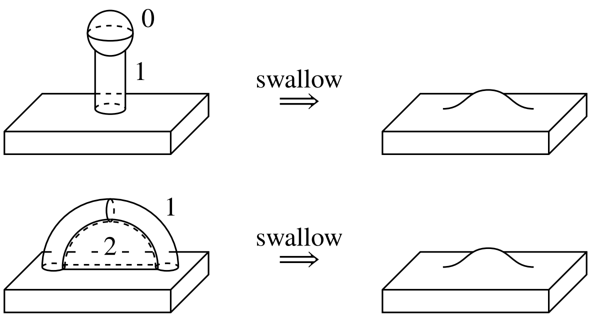
The other theorem allows handles to cancel.

Definition 1.8. The *belt* of the handle $D^\lambda \times D^{n-\lambda}$ is $\{0\} \times \partial D^{n-\lambda}$, and the *cobelt* is $\partial D^\lambda \times \{0\}$. They are subsets of the boundary of the handle. ◁

Theorem 1.9 (Cancellation). *Suppose that*

$$N \cup h^\lambda \cup h^{\lambda+1}$$

is a handlebody obtained from N by attaching two handles. If the cobelt of $h^{\lambda+1}$ intersects the belt of h^λ in exactly one point, then the new handlebody is diffeomorphic to N.



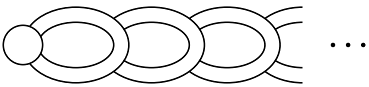
For the proofs of these theorems, see [Mat02] or [Mil65].

Handlebodies are ubiquitous in the following sense.

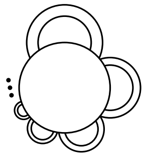
Theorem 1.10. *Every manifold is diffeomorphic to a good handlebody.*

Proof. [Wal16, Corollary 5.2.2]. □

However, some powerful tools, such as the rearrangement theorem, fail for infinite handlebodies. For example, consider the 2-dimensional handlebody depicted below.



If this handlebody were similar to a good handlebody, then infinitely many 1-handles would be attached on the 0-handle simultaneously, so that the local finiteness criterion would fail. In other words, if we attach handles in that way (pictured below), we would get a topological space that is not a manifold.



The purpose of this paper is to introduce a variant of the notion of handlebodies, which accepts this kind of spaces as handlebodies. We will prove the rearrangement and cancellation theorems for this kind of handlebodies. These results will make handle theory more available to non-compact manifolds.

2 Weak handlebodies

Here is the definition of our new notion of a handlebody.

Definition 2.1. A *weak relative n-handlebody* is a finite or infinite sequence

$$A = N_0 \subset N_1 \subset \cdots$$

of n-manifolds, such that each N_i is obtained from N_{i-1} by attaching (finitely or infinitely many) handles of the same type, i.e.,

$$N_i \simeq N_{i-1} \cup_{\Phi_i} \Big(\coprod_\alpha h^\lambda\Big),$$

where the attaching map

$$\Phi_i : \coprod_\alpha \big(\partial D^\lambda \times D^{n-\lambda}\big) \to \partial N_{i-1}$$

is required to be a closed embedding, which can be interpreted as *stepwise local finiteness*. Thus the space

$$N := \varinjlim (N_i \setminus \partial N_i)$$

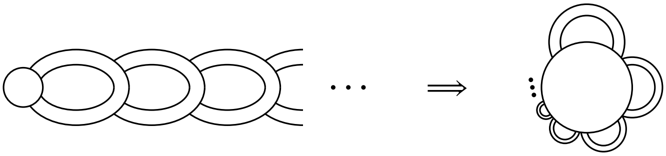
is a smooth manifold without boundary. ◁

By abuse of language, we say that (N, A) is a weak relative handlebody. If $A = \varnothing$, then we say that N is a *weak handlebody*.

The weak handlebody discards information about the boundary. However, this makes no difference when we talk about manifolds without boundary.

Moreover, it becomes possible for the rearrangement theorem to be true.

Theorem 2.2 (Rearrangement)**.** *Every weak relative handlebody is similar to a good weak relative handlebody.*

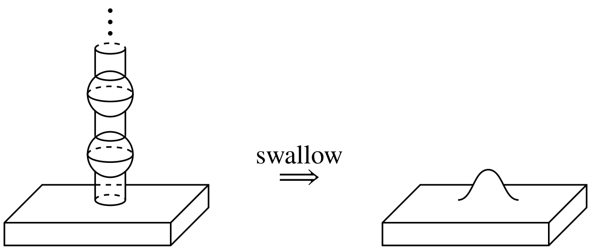


As before, two relative handlebodies are *similar*, if the pairs (N, A) are diffeomorphic as pairs of manifolds, and for every λ, they have the same number (possibly infinite) of λ-handles. A relative handlebody is *good*, if every λ-handle is attached on handles of type $< \lambda$, or on A.

The proof will be given in §4.

The cancellation theorem is also true, with some modifications.

Theorem 2.3 (Cancellation)**.** *If a weak handlebody has a collection of pairs of handles satisfying the condition for cancellation, then these pairs can be cancelled simultaneously.*



Precisely, *cancellation* means that the original handlebody is diffeomorphic to a new handlebody, with the number of handles cut down.

A more precise formulation and the proof will be given as (4.13).

3 Preliminaries on differential topology

This section will provide preliminary results that will be needed in the proofs of our main theorems. The reader is suggested to skip to §4 to read the proofs there, and come back for the results when they are used.

Lemma 3.1. *Let M, N be boundaryless manifolds, and $K \subset M$ a compact subset. If $f : M \to N$, and $f|_K$ is an injective local diffeomorphism, then there exists a neighbourhood U of K in M, such that $f|_U$ is a diffeomorphism onto its image.*

Proof. This is a standard exercise in mathematical analysis. ◻

Lemma 3.2. *Let M be a Riemannian n-manifold without boundary, and let S be a compact k-submanifold without boundary. Let $N(S)$ denote the normal bundle of S in M, and let $N_\varepsilon(S)$ denote the ε-neighbourhood of S in $N(S)$. There exists $\varepsilon > 0$ such that the map*

$$N_\varepsilon(S) \ni (p, v) \mapsto \exp_p(v)$$

is a diffeomorphism of $N_\varepsilon(S)$ onto its image.

Proof. By (3.1) and the inverse function theorem. □

Lemma 3.3. *Let M, N be manifolds with $\partial N = \varnothing$, and let $A \subset M$ be a closed subset. A smooth map $f : A \to N$ can be extended to M, if and only if it can be continuously extended to M.*

This is also true if $\partial N \neq \varnothing$ and $f(A) \subset \partial N$. In this case, if the continuous extension sends $M \setminus A$ into $N \setminus \partial N$, then we may require the smooth extension to have the same property.

Proof. [Lee12, Corollary 6.27]. The last statement follows from the construction in the proof of [Lee12, Theorem 6.21]. □

The next extension lemma will reduce our pain dealing with boundaries.

Lemma 3.4. *Let $M \subset \widetilde{M}$ be n-manifolds, $\partial\widetilde{M} = \varnothing$. For every closed submanifold $S \subset M$, there exists a submanifold $\widetilde{S} \subset \widetilde{M}$, such that $\partial\widetilde{S} = \varnothing$, $\dim S = \dim \widetilde{S}$, and $S \subset \widetilde{S}$. If moreover $\partial S \subset \partial M$, then we may require $S = \widetilde{S} \cap M$.*

Proof. Let $S^+ := S \cup \partial S \times [0, +\infty)$, with their boundaries identified in the obvious way. Choose a complete metric on S^+. By (3.3), the inclusion $S \hookrightarrow \widetilde{M}$ can be extended to a smooth map $f : S^+ \to \widetilde{M}$.

We embed $\widetilde{M}$ in some $\mathbb{R}^N$ as a closed submanifold. Let B_r denote the open ball in $\mathbb{R}^N$ centred at 0 and of radius r, or $\varnothing$ if $r \le 0$. For each $i \in \mathbb{N}$, denote $K_i := (\overline{B}_i \setminus B_{i-1}) \cap S$, and $U_i := (B_{i+1} \setminus \overline{B}_{i-2}) \cap \widetilde{M}$. By (3.1), for each i there exists $\varepsilon_i > 0$ so that f is a diffeomorphism of the ε_i-neighbourhood of $K_{i-2} \cup \cdots \cup K_{i+2}$ in S^+ onto its image (where $K_{-1} := K_0 := \varnothing$). We may assume that the images of the ε_i-neighbourhoods of $K_{i-2}, \ldots, K_{i+2}$ are contained in $U_{i-2}, \ldots, U_{i+2}$ respectively. Let $\varepsilon'_i := \min(\varepsilon_{i-2}, \ldots, \varepsilon_{i+2})$ (where $\varepsilon_{-1} := \varepsilon_0 := \varepsilon_1$), and let $\widetilde{S} := \bigcup_i (\varepsilon'_i\text{-neighbourhood of } K_i) \subset S^+$. Then $f|_{\widetilde{S}}$ is an injective local diffeomorphism, and thus a diffeomorphism onto its image.

For the last statement, it suffices to show that f can be chosen so that $f^{-1}(M) = S$. By the last statement of (3.3), we only need a continuous map $g : \partial S \times [0, +\infty) \to \widetilde{M}$ such that $g|_{\partial S}$ is the inclusion and $g^{-1}(\partial S) = \partial S \times 0$. This can be constructed as follows. Let $\varepsilon : \partial S \to \mathbb{R}_{>0}$ be a continuous function such that for all $p \in \partial S$, the exponential map $\exp_p(v)$ is defined for $|v| \le \varepsilon(p)$. Now g may be defined to be the map $(p, r) \mapsto \exp_p(\min(r, \varepsilon(p))\, \eta_p)$, where η_p is the unit normal vector of ∂M in $\mathrm{T}_p\widetilde{M}$, pointing outwards from M. □

Remark 3.5. The same proof applies if M and S have corners, with a modified construction of S^+. The last statement remains true if the corners of M are convex. ◁

Definition 3.6. Let M be a manifold, and let S, T be submanifolds. We say that S and T *intersect transversely*, or that S is *transverse to* T, if for all $p \in S \cap T$, we have $\mathrm{T}_p M = \mathrm{T}_p S + \mathrm{T}_p T$. Note that S, T does not necessarily intersect, and they are not necessarily boundaryless. ◁

Proposition 3.7. *If S is transverse to T, and if their boundaries are contained in ∂M, then $S \cap T$ is a submanifold of M.* □

Definition 3.8. Let $S_1, \dots, S_k \subset M$ be submanifolds of M. The set $\{S_1, \dots, S_k\}$ is said to be *transverse*, if when $k > 0$, $\{S_1, \dots, S_{k-1}\}$ is transverse (defined inductively on k), and S_k is transverse to everything in $\left\{ \bigcap_{i \in I} S_i \mid I \subset \{1, \dots, k - 1\} \right\}$. ◁

In this case, if we assume that $\partial S_i \subset \partial M$ for all i, then every possible intersection of the S_i is a submanifold.

The next lemma says that transverse submanifolds can be made orthogonal to each other by choosing an appropriate metric.

Lemma 3.9. *Let $S_1, \dots, S_k \subset M$ be transverse closed submanifolds. Then there exists a Riemannian metric on M, such that whenever $p \in S_i \cap S_j$, the subspaces $\mathrm{T}_p S_i$ and $\mathrm{T}_p S_j$ of $\mathrm{T}_p M$ are orthogonal.*

Proof. By (3.4), we may assume that M and the S_i are boundaryless.

For every $p \in M$, choose a neighbourhood U_p that does not intersect with any S_i not containing p, such that U_p is diffeomorphic to $\mathbb{R}^n$, and under this diffeomorphism, the S_i containing p (if any) are identified with some coordinate subspaces of $\mathbb{R}^n$. This is possible since we may construct smooth functions $f_1, \dots, f_n$ near p, such that each of these S_i is locally cut out by the zeros of some of these f_j (each f_j is used by at most one S_i), and such that $\mathrm{d}f_1, \dots, \mathrm{d}f_n$ form a basis for $\mathrm{T}_p^* M$. Using these f_j as local coordinate functions, we get a desired diffeomorphism by the inverse function theorem.

Let g_p denote the metric on U_p induced from $\mathbb{R}^n$. Let $\{\rho_p\}$ be a partition of unity subordinate to the open cover $\{U_p\}$. Then $\sum_p \rho_p g_p$ is a desired metric. □

One important result in intersection theory is the transversality theorem [GP74, p. 68]. Here is one of its important corollaries.

Lemma 3.10. *Let M be a boundaryless n-manifold, and let $S_1, \dots, S_k, T$ be submanifolds, where $\partial T = \varnothing$, but the S_i may have corners. For any neighbourhood U of T, T can be moved by an isotopy within U, to become transverse to all S_i.*

Moreover, the isotopy can be taken to be fixed on any closed subset of T where T is already transverse to all the S_i.

Proof. By (3.4), we may assume that the S_i are boundaryless. See [GP74, p. 70] for a proof in this case. □

The next few results are concerned with the existence of functions with certain properties. The gradient vector fields of these functions will be very useful. We will use the flow of these vector fields on the boundary of a handlebody, to manipulate the handles attached on it.

Lemma 3.11. *Let M be a closed n-manifold, and let $S_1, \dots, S_k \subset M$ be transverse compact submanifolds without boundary. Then there exists a smooth function $f : M \to [0,1]$, such that*

(i) $f^{-1}(0) = S := S_1 \cup \cdots \cup S_k$.

(ii) *f has no critical values other than* $0, 1$.

Proof. By (3.9), there exists a Riemannian metric on M, such that any two of $S_1, \dots, S_k$ intersect orthogonally.

We claim that for small enough $\varepsilon > 0$ and $i = 1, \dots, k$, there exists a smooth function $f_{i,\varepsilon} : M \to [0,1]$, such that

(i) $f_{i,\varepsilon}^{-1}(0) = S_i$.

(ii) Whenever $f_{i,\varepsilon}(p) \in (0,1)$, we have $\nabla f_{i,\varepsilon}(p) \neq 0$, and $\nabla f_{i,\varepsilon}$ points orthogonally outwards from S_i, in the sense that its flow line through p is the unique minimising geodesic from p to S_i.

Namely, by (3.2), we may define a function $d_{i,\varepsilon}$ to be $1/\varepsilon$ times the distance to S_i, and let $f_{i,\varepsilon} = \rho \circ d_{i,\varepsilon}$, where $\rho : [0,+\infty) \to [0,1]$ is a smooth function, such that $\rho(r) = 1$ when $r \geq 1$, $\rho(0) = \rho'(0) = \cdots = 0$, and $\rho'(r) > 0$ if $0 < r < 1$.

Denote $U_{i,\varepsilon} = f_{i,\varepsilon}^{-1}([0,1))$. Now we take $f = f_{1,\varepsilon} \cdots f_{k,\varepsilon}$, where we choose $\delta > \varepsilon > 0$ small enough, so that

(i) $f_{i,2\delta}$ is defined for all i.

(ii) For every path of length ℓ in $U_{i,2\delta}$, its projection to S_i has length $< 2\ell$.

(iii) Whenever $p \in \bigcap_{i \in I} U_{i,\varepsilon}$ for some $I \subset \{1, \dots, k\}$, there exists a path from p to $\bigcap_{i \in I} S_i$ with length $< \delta$.

(iv) Whenever a vector is parallel transported along a loop of length $< 4\delta$, its direction changes by an angle $< \pi/8$.

(v) Whenever a vector is parallel transported along a curve of length $< 2\delta$ on S_i, the angle between the vector and S_i changes by $< \pi/8$.

(iii) is possible, since if no such ε exists, we would have a contradiction against sequential compactness of S. (iv) is possible since the Riemann curvature tensor of M is bounded. (v) is possible since the second fundamental forms of the S_i are bounded.

We claim that for any p that is ε-near S, p is not a critical point of f. (For simplicity, two things are *ε-near* if their distance is $< \varepsilon$.) Indeed, suppose p is ε-near S_i for $i \in I$, while not ε-near the others. Denote $S_I := \bigcap_{i \in I} S_i$ and $U_{I,\varepsilon} := \bigcap_{i \in I} U_{i,\varepsilon}$. Note that

$$\nabla f(p) = \sum_{i \in I} \frac{f(p)}{f_{i,\varepsilon}(p)} \nabla f_{i,\varepsilon}(p).$$

Let p_i denote the projection of p to S_i, and let q be a point in S_I that is δ-near p. Let γ be a path from p to q of length $< \delta$ (thus it falls in $U_{I,2\delta}$), and let γ_i be the projection of γ to S_i, which is a path from p_i to q of length $< 2\delta$. Let $v_i \in \mathrm{T}_{p_i} M$ be the vector pointing to p, of length $|\nabla f_{i,\varepsilon}(p)|$. Let $w_i \in \mathrm{T}_q M$ denote its parallel transport along γ_i. Then the angle between w_i and $\mathrm{T}_q S_i$ is $> 3\pi/8$. Note that w_i is obtained from $\nabla f_{i,\varepsilon}(p) \in \mathrm{T}_p M$ by parallel translation along the path p—p_i—q. If we instead translate it directly along γ, the resulting vector, say u_i, would have a direction that differ from w_i by $< \pi/8$. Thus the angle between u_i and $\mathrm{T}_p S_i$ is $> \pi/4$. But the S_i are transverse and orthogonal at p, hence by a standard argument in linear algebra, the u_i are linearly independent, and so are the vectors $\nabla f_{i,\varepsilon}(p)$. This implies that $\nabla f(p) \neq 0$. □

Remark 3.12. This is not true for an arbitrary closed subset S. For example, consider $S =$ Cantor set $\subset M = S^1$. ◁

We will need a slightly more general form of this lemma, which follows from exactly the same construction (involving distance functions to submanifolds with boundary, which are not necessarily smooth near the boundary).

Corollary 3.13. *Let M be a closed n-manifold, and let $S_1, \dots, S_k \subset M$ be transverse compact submanifolds, possibly with boundary. Let U denote an open neighbourhood of $\partial S_1 \cup \cdots \cup \partial S_k$. Then there exists a smooth function $f : M \setminus U \to [0,1]$, such that*

(i) $f^{-1}(0) = (S_1 \cup \cdots \cup S_k) \setminus U$.

(ii) *f has no critical values other than* $0, 1$. □

Finally, we mention two results concerning isotopies.

Lemma 3.14 (Isotopy extension theorem)**.** *If $\partial M = \varnothing$ and S is compact, then every compactly supported isotopy $h : S \times \mathbb{R} \to M$ can be extended to a compactly supported diffeotopy on M.*

Proof. [Hir76, Theorem 8.1.3]. □

Lemma 3.15. *Let M be a connected n-manifold. If M is orientable, then two embeddings $D^n \hookrightarrow M$ are isotopic if and only if they have the same orientation. If M is non-orientable, then all such embeddings are isotopic.*

Proof. We first consider the special case $M = \mathbb{R}^n$. We may assume the embedding $f : D^n \to \mathbb{R}^n$ sends 0 to 0, preserving orientation. Using a smooth path in $\mathrm{GL}(n)$, i.e. a smooth family of linear transformations, we may assume $\mathrm{d}f(0) = \mathbb{1}$. We further shrink the disk linearly (w.r.t. the disk) by an isotopy, and then enlarge it linearly (w.r.t. $\mathbb{R}^n$) to recover $\mathrm{d}f(0) = \mathbb{1}$, so that if we denote $g(x) := f(x) - x$, then the norm of $\mathrm{d}g(x)$ (supremum of $|\mathrm{d}g(x)v|/|v|$ for $v \neq 0$) is $< 1/2$ for all $x \in D^n$. This will ensure that the linear homotopy h_t from f to the standard inclusion $D^n \hookrightarrow \mathbb{R}^n$ is injective for all t, and thus it is an isotopy.

For the general case, we may shrink D^n into a coordinate chart, and then transport it along a finite sequence of coordinate charts. Doing this for both disks, we are reduced to the first case. □

4 The proofs

This section is dedicated to the proofs of the rearrangement theorem (2.2) and the cancellation theorem (2.3) for weak handlebodies.

The rearrangement theorem

Definition 4.1. Let M be a manifold. A smooth function $\theta : M \to [0,1]$ is called a *level function*, if $\theta^{-1}(0) = \partial M$ and θ has no critical points on ∂M. ◁

Proposition 4.2. *Every smooth manifold has a level function.*

Proof. Exercise for the reader. □

Definition 4.3. Let M be a manifold with a level function θ, and let $\varepsilon > 0$. An *ε-sliding* of M is a diffeotopy $h : M \times \mathbb{R} \to M$, such that

(i) h is supported in $\{p \in M \mid \theta(p) < \varepsilon\}$.

(ii) θ is preserved by h. ◁

Intuitively, an ε-sliding moves the boundary around, and keeps most of the interior fixed. When we talk about an ε-sliding, we will implicitly assume that some level function θ is chosen.

Lemma 4.4. *Let M be a manifold, and let $\varepsilon > 0$. If X is a compactly supported vector field on ∂M, then X can be extended to a vector field $\widetilde{X}$ on M, which generates an ε-sliding of M.*

Proof. Let $U := \{p \in M \mid \theta(p) < \varepsilon\}$, where θ is the level function. Let $K \subset \partial M$ denote the support of X. Let $\partial M \times [0, +\infty) \subset M$ be a collar neighbourhood. Since K is compact, we may assume $K \times [0, \varepsilon] \subset U$. We also assume that θ has no critical points in $K \times [0, \varepsilon]$.

Let $\rho : [0, +\infty) \to [0, 1]$ be a smooth function such that $\rho(0) = 1$ and $\rho(r) = 0$ when $r > \varepsilon/2$. We first extend X to M by letting

$$X_p := \begin{cases} \rho(r)\, X_q, & p = (q, r) \in K \times [0, \varepsilon], \\ 0, & \text{otherwise.} \end{cases}$$

Next we take a metric on M, so that for all $p \in \partial M$, $\nabla\theta(p)$ is orthogonal to $\mathrm{T}_p(\partial M) \subset \mathrm{T}_p M$. This is possible by the constant rank theorem, together with a partition of unity. We define

$$\widetilde{X} := X - \frac{\langle X, \nabla\theta \rangle}{|\nabla\theta|^2} \nabla\theta.$$

Since $\widetilde{X}$ is supported in $K \times [0, \varepsilon]$, it generates a diffeotopy h. Since $\widetilde{X}$ is orthogonal to $\nabla\theta$, h is indeed an ε-sliding. □

Remark 4.5. If X is instead supported in a disjoint union of countably many compact sets, the lemma still holds true. The reason is that the collar is global, and will make sure that the sets $K \times [0, \varepsilon]$ are disjoint.

Corollary 4.6. *Every diffeotopy of ∂M supported in a disjoint union of* (possibly infinitely many) *compact sets can be extended to an ε-sliding of M.*

Proof. Apply (4.4) to the velocity field of the diffeotopy. This is a time-dependent vector field, but the above proof applies as well. □

Lemma 4.7. *Let (N, A) be a relative n-handlebody, and let $\Phi : \partial_1 h^\lambda \to \partial N$ be an embedding, where $\lambda \neq 0, n$. For any neighbourhood U of the cobelt $\Phi(\partial D^\lambda \times 0)$ in ∂N, the image of Φ can be moved into U by an isotopy.*

Proof. Take any metric on ∂N. Then there exists $\varepsilon > 0$ such that $\Phi(\partial D^\lambda \times D^{n-\lambda}_\varepsilon) \subset U$, where $D^{n-\lambda}_\varepsilon$ denotes the disk of radius ε. Now shrink $\Phi(\partial D^\lambda \times D^{n-\lambda})$ into this ε-neighbourhood linearly. □

Definition 4.8. Let (N, A) be a weak relative n-handlebody, let $i \geq 0$ be a fixed integer, and let $h : N_i \to N_i$ be a diffeomorphism. The *perturbation by h* of (N, A) is a relative handlebody (N^h, A), with attaching maps Φ^h_j, defined as follows:

- $N^h_j = N_j$ and $\Phi^h_j = \Phi_j$ for all $j \leq i$.
- $\Phi^h_{i+1} = h \circ \Phi_{i+1}$. This induces a diffeomorphism $h_{i+1} : N_{i+1} \to N^h_{i+1}$.
- $\Phi^h_{i+2} = h_{i+1} \circ \Phi_{i+2}$, inducing a diffeomorphism $h_{i+2} : N_{i+2} \to N^h_{i+2}$.
- And so on.

The perturbation is called an *ε-perturbation* if h is an ε-sliding.

A finite composition of perturbations is also considered a perturbation. ◁

After a perturbation, the handles attached after the i-th step will get attached to different handles, but the new handlebody is similar to the original one. We will see that every handlebody can be perturbed into a good one.

Notation 4.9. Write

$$\partial_1 h^\lambda := \partial D^\lambda \times D^{n-\lambda} \quad \text{and} \quad \partial_2 h^\lambda := D^\lambda \times \partial D^{n-\lambda},$$

so that $\partial h^\lambda = \partial_1 h^\lambda \cup \partial_2 h^\lambda$. ◁

The idea of our proof of the rearrangement theorem is to create a vector field on the side of each handle, so that other handles attached on it can be slid out of it along this vector field.

Definition 4.10. Let h^λ be a λ-handle, where $\lambda \neq 0, n$. Let $B \subset \partial_2 h^\lambda$ be a closed subset, and let O denote the belt of h^λ. An *expanding field* of h^λ with respect to B is a vector field X on $\partial_2 h^\lambda$, such that

- $X_p = 0$ if and only if $p \in B \cup O$.

- For every $p \in \partial_2 h^\lambda \setminus (B \cup O)$, the flow line of X starting from p reaches $\partial\partial_2 h^\lambda$ within finite time.

Let (N, A) be a finite relative n-handlebody. Its *skeleton* is defined to be the set Σ of compact submanifolds of ∂N, consisting of

(i) The intersection of ∂N with the boundary of the image of every attaching map.

(ii) The intersection of ∂N with the belt of each handle.

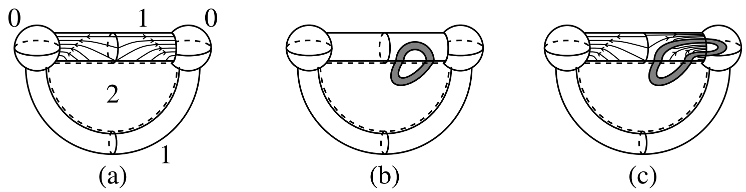
(iii) All possible intersections of the manifolds listed above.

In order for (iii) to be well-defined, we require that the set of submanifolds in (i) and (ii) is transverse. Such a finite relative handlebody is said to be *regular*. We denote by Σ_i the skeleton of (N_i, A), and call it the *i-skeleton* of (N, A).

A regular n-handlebody (N, A) is said to be *expanded*, if every λ-handle h^λ with $\lambda \neq 0, n$ has an expanding field with respect to the union of images of attaching maps of handles that are attached after it. ◁

An expanding field does not necessarily exist for arbitrary B. For example, consider the case $n = 2$, $\lambda = 1$ and B is a point not in O.

The next lemma makes clear how expanding fields work. The main idea of the proof is visualised below.



(a) A 4-handlebody, where numbers indicate types of handles. A 1-handle is drawn with its expanding field. (Of course what is seen here is a 3-dimensional analogue of the real thing.)
(b) Attach a 3-handle, where the shaded part indicates the image of the attaching map.
(c) Stretch the attaching map using the expanding field, and construct a new expanding field on the 1-handle.

Lemma 4.11. *Let (N, A) be a finite relative n-handlebody with i handles, such that (N_{i-1}, A) is good and expanded. Then N_{i-1} can be ε-slid, so that the induced ε-perturbation is good and expanded.*

Proof. Let h^λ denote the handle to be attached, and let Φ denote the attaching map. Let h_j denote the handle attached in the j-th step. Since (N_{i-1}, A) is good, we may assume $h_1, \ldots, h_{i-1}$ are of ascending types.

First we try to slide $\mathrm{im}(\Phi)$ to avoid all handles of type $\geq \lambda$. Suppose $\mathrm{im}(\Phi)$ intersects with some h_j with type $\geq \lambda$, and we fix the largest j with this property.

Then $\mathrm{im}(\Phi)$ does not intersect with $h_{j+1}, \ldots, h_{i-1}$. By reparametrising h^λ, we may assume that the cobelt $\Phi(\partial D^\lambda \times 0)$ avoids the belt O of h_j. By (4.7), we can shrink $\mathrm{im}(\Phi)$ so that it does not meet O. We then extend the expanding field on h_j to a small neighbourhood in $\partial N_j \cap \partial N_{i-1}$ that does not meet any handles attached after the j-th step, and we could slide $\mathrm{im}(\Phi)$ out of $\partial_2 h_j$ using the flow φ_t of the expanding field for large enough t. We do this for all h_j, in descending order until $\mathrm{im}(\Phi)$ finally gets to a (literally) good position, i.e. within $\partial N_{i'-1}$, where i' denotes the step where the first handle of type $\geq \lambda$ is attached.

Our next plan is to do an ε-sliding for each $j = i' - 1, \ldots, 1$, making the corresponding handle expanded at each time. By (3.10), we may assume that $\Phi(\partial\partial_2 h^\lambda)$ is transverse to everything in $\Sigma_{i'-1}$, while keeping $\mathrm{im}(\Phi)$ disjoint with $h_{i'}, \ldots, h_{i-1}$.

In the j-th step, let X denote the expanding field on $\partial_2 h_j$. Let $B \subset \partial N_j$ denote the union of images of attaching maps in steps $j+1, \ldots, i-1$, intersected with ∂N_j. Let Φ' denote the composition of Φ with the previously executed slidings.

Let $S_{j+1}, \ldots, S_i$ be the boundaries of images of attaching maps in the corresponding steps, intersected with ∂N_j, so that $S_i = \Phi'(\partial\partial_2 h^\lambda) \cap \partial N_j$. Let O denote the belt of h_j. Thus the S_k and B are submanifolds possibly with corners, and the corners of B are concave. By (3.4) and the remark after it, we may extend the S_k that have boundaries a little bit into B, so that the extended manifolds S'_k are compact, and $\partial S'_k \subset B^\circ$ for all k.

By regularity of (N_{i-1}, A), we may assume $O, S'_{j+1}, \ldots, S'_i$ are transverse (shrinking a bit the extended part if necessary). Thus by (3.13), there is a smooth function $f : \partial N_j \setminus B^\circ \to [0,1]$, with $f^{-1}(0) = (O \cup S'_{j+1} \cup \cdots \cup S'_i) \setminus B^\circ$, with no critical values other than $0, 1$. We modify f by letting $f|_B \equiv 0$ and $f|_{\mathrm{im}(\Phi')} \equiv 0$. Now f becomes a smooth function $\partial N_j \to [0,1]$ with $f^{-1}(0) = O \cup B \cup \mathrm{im}(\Phi')$, and it has no critical values other than $0, 1$.

We extend X to a compactly supported smooth vector field on ∂N_j, so that $X|_B = 0$, and for any $p \in \partial\partial_2 h_j$ such that $X_p \neq 0$, the change of f along the extended part of the flow line of X through p is $< 1/2$.

We take a large number $t \in \mathbb{R}$, such that the flow $h := \varphi_t$ of X satisfies

$$h\big(f^{-1}([1/2, 1]) \cap \partial_2 h_j\big) \cap \partial_2 h_j = \varnothing.$$

By (3.10), after performing a small isotopy preserving the above condition, keeping B fixed, we may assume that $h(\Phi'(\partial\partial_2 h^\lambda))$ is transverse to everything in Σ_{i-1}.

Take a metric on $\partial_2 h_j$, and put

$$Y_p := \mathrm{d}h(\nabla f(h^{-1}(p))).$$

One verifies that Y is a new expanding field on $\partial_2 h_j$, but with respect to $B \cup \mathrm{im}(h \circ \Phi')$ instead of B. Note that h keeps B fixed. Finally we execute the ε-sliding induced by h.

We have finished the j-th step. Running over $j = i' - 1, \ldots, 1$ gives a desired ε-sliding, and gives desired expanding fields on all handles except the i-th one. But the i-th handle is trivially done. Regularity is preserved throughout the above construction. □

Having done much preparatory work, we are finally ready to take our final step to proving (2.2), namely, the concatenation of these infinitely many slidings given by the preceding lemma.

Proof of (2.2). Let (N, A) be a weak relative n-handlebody. We may assume that only one handle is attached in each step. For each $i \geq 0$, we may regard (N_i, A) as a non-weak relative handlebody. Let θ_i be a level function on N_i for $i = 0, 1, \dots$, so that $\theta_i \leq \theta_{i+1}$ for all i.

Denote $N^1 = N$. Note that (N_1^1, A) is automatically expanded. By (4.11), N_1^1 can be $1/2$-slid, so that if we denote by (N^2, A) the induced perturbation, then (N_2^2, A) is expanded. We continue to $1/4$-slide N_2^2 to get a perturbation (N^3, A) of (N^2, A), so that (N_3^3, A) is expanded. Continuing this process, using $1/2^i$ in the i-th step, we get a commutative grid of maps

$$\begin{array}{ccccccccccc}
N_0^1 & \hookrightarrow & N_1^1 & \hookrightarrow & N_2^1 & \hookrightarrow & N_3^1 & \hookrightarrow & N_4^1 & \hookrightarrow & \cdots \\
 & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\
N_0^2 & \hookrightarrow & N_1^2 & \hookrightarrow & N_2^2 & \hookrightarrow & N_3^2 & \hookrightarrow & N_4^2 & \hookrightarrow & \cdots \\
 & & & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\
N_0^3 & \hookrightarrow & N_1^3 & \hookrightarrow & N_2^3 & \hookrightarrow & N_3^3 & \hookrightarrow & N_4^3 & \hookrightarrow & \cdots, \\
 & & & & & & \downarrow\simeq & & \downarrow\simeq & & \\
 & & & & & & \cdots & & \cdots & &
\end{array}$$

where the vertical maps are all diffeomorphisms. By construction N_i^i and N_i^{i+1} are naturally identified, but note that the identification is not the diffeomorphism (sliding) in the diagram. We define θ_i on N_j^i for $i \leq j + 1$ to be the level function inherited from $N_j = N_j^1$. Thus for N_i^i and N_i^{i+1}, their functions θ_i are the same under the natural identification.

We define a good weak relative n-handlebody (N', A) by $N_i' := N_i^i$, and the inclusion maps being $N_i^i = N_i^{i+1} \hookrightarrow N_{i+1}^{i+1}$. This is *not* the map in the above diagram! But for $p \in N_i^i \subset N_{i+1}^{i+1}$, we still have $\theta_i(p) \leq \theta_{i+1}(p)$. We will show that N is diffeomorphic to N'.

Consider the (not commutative) diagram

$$\begin{array}{ccccccc}
N_1 & \hookrightarrow & N_2 & \hookrightarrow & N_3 & \hookrightarrow & \cdots \\
\| & & \downarrow\simeq & & \downarrow\simeq & & \\
N_1^1 & \hookrightarrow & N_2^2 & \hookrightarrow & N_3^3 & \hookrightarrow & \cdots,
\end{array} \tag{$*$}$$

where the vertical maps are as in the above grid-like diagram. By comparing the above two diagrams, the square

$$\begin{array}{ccc}
N_i & \hookrightarrow & N_{i+1} \\
\downarrow\simeq & & \downarrow\simeq \\
N_i^i & \hookrightarrow & N_{i+1}^{i+1}
\end{array}$$

commutes for some $p \in N_i$ if and only if p is fixed by the sliding $N_i^i \to N_i^{i+1}$. This happens whenever $\theta_i(p) > 1/2^i$. Thus for

$$U_i := \{p \in N_i \mid \theta_i(p) > 1/2^i\},$$

the diagram $(*)$ commutes after the i-th square. Thus we have an induced diffeomorphism of U_i onto a subset of N'. Since $N = \bigcup_{i=1}^{\infty} U_i$, we have an induced map $\varphi : N \to N'$ which is a diffeomorphism onto its image. Clearly φ is also surjective since $N' = \bigcup_{i=1}^{\infty} \operatorname{im}(U_i \to N')$. Thus φ is a diffeomorphism. □

The cancellation theorem

We first state the cancellation theorem for finite handlebodies.

Theorem 4.12 (Cancellation, finite case)**.** *Let $N = A \cup_{\Phi_1} h^\lambda \cup_{\Phi_2} h^{\lambda+1}$ be a relative handlebody with* 2 *handles. If the cobelt of $h^{\lambda+1}$ and the belt of h^λ intersect transversely at precisely one point, then N is diffeomorphic to A. This diffeomorphism can be taken to be fixed on any closed subset of $A \setminus \operatorname{im}(\Phi_1) \setminus \operatorname{im}(\Phi_2)$.*

Proof. [Wal16, Theorem 5.4.3] or [Mat02, Theorem 3.34]. □

For convenience, we introduce the following terminology. In a good relative handlebody, a pair of λ- and $(\lambda+1)$-handles is called a *cancelling pair*, if the cobelt of the $(\lambda+1)$-handle and the belt of the λ-handle intersect transversely at precisely one point. The $(\lambda+1)$-handle will be called the *canceller*, and the λ-handle will be called the *cancellee*.

We say that we can *cancel* a set S of handles from a weak relative handlebody (N, A), if N is diffeomorphic to a weak relative handlebody, whose λ-handles correspond to the λ-handles of N that are not in S, and the combinatorics concerning which handles are attached to other handles should not change, except that a handle attached to a cancelled handle may get attached to a handle which a cancelled handle was attached to.

Theorem 4.13 (Cancellation, infinite case)**.** *Let (N, A) be a weak relative handlebody whose handles are attached one at a time, and let S be a set of handles consisting of cancelling pairs that do not have handles in common, such that every canceller in S is attached immediately after its cancellee. Then S can be cancelled.*

Proof. We will construct a new weak handlebody N' diffeomorphic to N, with the desired property.

Let $N_0' := N_0 = A$, and let θ_0 be a level function on it. Let $\varphi_0 : N_0 \to N_0'$ be a diffeomorphism. As we construct N', we will construct diffeomorphisms $\varphi_i : N_i \to N_i'$, and level functions θ_i on N_i', such that $\theta_i \le \theta_{i+1}$ on N_i'. This will be implicit in the below construction.

In the i-th step, if the i-th handle of N is not a canceller or a cancellee, then we attach a same handle to N' according to φ_{i-1}, and let φ_i be the induced diffeomorphism. If it is a cancellee (followed by its canceller), then we will not attach handles to N' in the i-th and $(i+1)$-th steps. Instead, we construct by (4.12) a diffeomorphism φ_{i+1} to form a (not commutative) diagram

$$\begin{array}{ccccc} N_{i-1} & \hookrightarrow & N_i & \hookrightarrow & N_{i+1} \\ {\scriptstyle \varphi_{i-1}} \downarrow \simeq & & & & {\scriptstyle \varphi_{i+1}} \downarrow \simeq \\ N'_{i-1} & = & N'_i & = & N'_{i+1}. \end{array}$$

By the last statement of (4.12), we may require the diagram to commute in $U'_{i-1} := \{p \in N'_{i-1} \mid \theta_{i-1}(p) > 1/2^{i-1}\}$. We take a new level function θ_{i+1} on N'_{i+1}, so that for all $p \in N'_{i-1}$, we have

$$\theta_{i+1}(p) \geq \theta_{i-1}(p) \quad \text{and} \quad \theta_{i+1}(p) \geq \theta_{i-1}(\varphi_{i+1} \circ \varphi_{i-1}^{-1}(p)).$$

By a same argument as before, we have a colimit map from N' to N, which is an embedding. It is surjective by the choice of the functions θ_i. □

We have now completed the proof of our two main theorems. They can be combined to obtain the following result, which simplifies weak handlebodies.

Definition 4.14. Let $n \geq m \geq 0$ be integers. An (n, m)-*handlebody* is an n-handlebody whose handles are of type $0, 1, \ldots, m$. ◁

Corollary 4.15. *Every connected finite handlebody is diffeomorphic to a handlebody with only one* 0*-handle. Every connected weak handlebody is diffeomorphic to a good weak handlebody with only one* 0*-handle.*

Proof. For the finite case, we may suppose that the handlebody is good by (1.5). Thus it suffices to prove the statement for an $(n, 1)$-handlebody. Consider a graph with vertices corresponding to the 0-handles and edges corresponding to the 1-handles. Then this graph is connected. Taking a maximal tree of this graph, we may assume the handlebody is simply connected (by discarding the 1-handles not in the tree). Thus the result follows from (4.12), by cancelling the 0- and 1-handles a pair at a time.

For the weak case, we suppose that the handlebody is good by (2.2). We construct a graph and take a maximal tree in the same way, and the statement follows from (4.13). □

5 Handle chain complexes

One might notice the resemblance between a handlebody and a CW complex. In fact, they are related in the following way.

Definition 5.1. Let (N, A) be a good, weak or non-weak, relative handlebody. We define its *associated CW complex* (X, A) as follows. Assume for N that the handles are attached one at a time.

- Let $X_0 := A$ as a trivial relative CW complex over A, and let $p_0 : N_0 \to X_0$ be the identity map.
- If $N_1 = N_0 \cup_{\Phi_1} h^\lambda$, we define $X_1 := X_0 \cup_{\varphi_1} D^\lambda$, where $\varphi_1 := p_0 \circ \Phi_1|_{\partial D^\lambda \times 0}$. We define a continuous map $p_1 : N_1 \to X_1$ as follows. Let p_1 agree with p_0 on N_0, and let it be the projection to the core $D^\lambda \times 0$ of h^λ.
- We then define X_2 and p_2, and so on.

Thus $X := \varinjlim X_i$ is a relative CW complex over A. ◁

Proposition 5.2. *The associated CW complex is homotopy equivalent to the original, weak or non-weak, relative handlebody.*

Proof. The weak case needs an extra first step. Note that N_i is homotopy equivalent to $N_i \setminus \partial N_i$, since if we take a collar neighbourhood $\partial N_i \times [0, +\infty)$, then both spaces retract to $N_i \setminus \partial N_i \times [0, 1)$.

We claim that each p_i is a homotopy equivalence. This is shown by induction on i. Since the projection onto the core is a homotopy equivalence between h^λ and D^λ, by [MP12, Lemma 2.1.3], $N_i := N_{i-1} \cup_{\Phi_i} h^\lambda$ and $X_i := X_{i-1} \cup_{\varphi_i} D^\lambda$ are homotopy equivalent through the induced map p_i.

Finally, by [MP12, Lemma 2.1.10], the colimit map $p : N \to X$ is a homotopy equivalence. □

As a consequence, the (co)homology groups of a handlebody can be computed by the cellular (co)chain complex of its associated CW complex. We shall define an analogous (co)chain complex associated to a handlebody.

Definition 5.3. Let M, S, T be oriented m-, s- and t-manifolds without boundary, with S, T compact and $s + t = m$. Let $f : S \to M$, $g : T \to M$ be embeddings. If $f(S)$ is transverse to $g(T)$, then we define the *intersection number* of f, g to be

$$\#(f, g) := \sum_{p \in f(S) \cap g(T)} \pm 1,$$

where the sign is decided by whether the vector space isomorphism $\mathrm{T}_p M \simeq \mathrm{T}_p f(S) \oplus \mathrm{T}_p g(T)$ preserves $(+1)$ or reverses (-1) the orientation. ◁

Proposition 5.4. *Under the above assumptions, if f is isotopic to f' and g is isotopic to g', such that $f'(S)$ is transverse to $g'(T)$, then $\#(f, g) = \#(f', g')$.*

Proof. By (3.14), we may extend the isotopy from g to g' to an isotopy of M. Thus we may assume $g = g'$. For a proof in this case, see [GP74, p. 108]. □

Therefore, the intersection number is well-defined for isotopy classes of maps. This allows us to define the intersection number for submanifolds that are not necessarily transverse.

Definition 5.5. Let M, S, T be oriented m-, s- and t-manifolds without boundary, where S, T are compact submanifolds of M with $s + t = m$. Their *intersection number* $\#(S, T)$ is defined to be any $\#(f, g)$, such that f is isotopic to the embedding $S \hookrightarrow M$, g is isotopic to $T \hookrightarrow M$, and $f(S)$ is transverse to $g(T)$. ◁

For any given M, S, T, such f, g always exist. This is a consequence of (3.10).

Definition 5.6. Let N be a good, weak or non-weak, n-handlebody, and let G be an abelian group. The *handle chain complex* $\boldsymbol{C}_\bullet(N;G)$ of N is defined as follows.

- $C_\lambda(N;G) := \bigoplus_\alpha G \cdot [h_\alpha^\lambda]$ for $\lambda = 0, \dots, n$, and 0 otherwise. Here h_α^λ runs through all λ-handles of N.
- $\partial_\lambda : C_\lambda(N;G) \to C_{\lambda-1}(N;G)$ is defined on basis elements by
$$\partial_\lambda[h_\alpha^\lambda] := \sum \#\big(\text{cobelt}(h_\alpha^\lambda),\ \text{belt}(h_\beta^{\lambda-1})\big)\,[h_\beta^{\lambda-1}],$$
where the sum is taken over all $(\lambda-1)$-handles $h_\beta^{\lambda-1}$ attached before h_α^λ.

If N is locally finite, then we define the *handle cochain complex* $\boldsymbol{C}^\bullet(N;G)$ as follows.

- $C^\lambda(N;G) := C_\lambda(N;G)$.
- $d^\lambda : C^\lambda(N;G) \to C^{\lambda+1}(N;G)$ is defined on basis elements by
$$d^\lambda[h_\alpha^\lambda] := \sum \#\big(\text{belt}(h_\alpha^\lambda),\ \text{cobelt}(h_\beta^{\lambda+1})\big)\,[h_\beta^{\lambda+1}],$$
where the sum is taken over all $(\lambda+1)$-handles $h_\beta^{\lambda+1}$ attached after h_α^λ.

The relative versions, namely $\boldsymbol{C}_\bullet(N,A;G)$ and $\boldsymbol{C}^\bullet(N,A;G)$, are defined similarly. ◁

We need to prove that these are indeed chain complexes. We do this by showing that the homological version is isomorphic to the cellular chain complex of a CW complex.

Proposition 5.7. *Let X be the associated CW complex of N. If N is orientable or $2G = 0$, then $\boldsymbol{C}_\bullet(N;G)$ is isomorphic to the cellular chain complex $\boldsymbol{C}_\bullet^{\text{cell}}(X;G)$.*

Proof. Note that under the map $p_i : N_i \to X_i$, the intersection number corresponds to the sum of local degrees of the attaching map on the inverse image of $0 \in D^{\lambda-1}$. In the orientable case the orientation of each ∂N_i may be chosen to be compatible with the handles, so that degrees are counted correctly; otherwise they are only correct modulo 2. □

Proposition 5.8. *Suppose N is orientable or $2G = 0$. Then*

- *The chain complex $\boldsymbol{C}_\bullet(N;G)$ computes the singular homology $\boldsymbol{H}_\bullet(N;G)$.*
- *The cochain complex $\boldsymbol{C}^\bullet(N;G)$, if defined, computes the singular cohomology with compact support $\boldsymbol{H}_\text{c}^\bullet(N;G)$.*

Proof. The first statement is immediate; we prove the second one.

If N is compact (that is, finite), then $\boldsymbol{C}^\bullet(N;G)$ is precisely the dual of $\boldsymbol{C}_\bullet(N;G)$, i.e. obtained by applying $\text{Hom}_G(-,G)$. By [Hat01, Theorem 3.5], the dual of the cellular complex computes the singular cohomology. Thus $\boldsymbol{C}^\bullet(N;G)$

also computes the singular cohomology, which is equal to singular cohomology with compact support.

In the general case, suppose that the handles in N are attached one at a time. This does not affect the chain complexes. We use the fact that

$$H_{\mathrm{c}}^{\bullet}(X;G) \simeq \varinjlim_{K \text{ compact}} H^{\bullet}(X, X \setminus K; G)$$

for any space X [Hat01, below 3.33]. By local finiteness, for any compact $K \subset N$, there exists i such that $K \subset$ (interior of N_i in N). By excision we have $H^{\bullet}(N, N \setminus K; G) \simeq H^{\bullet}(N_i, N_i \setminus K; G)$. Thus

$$H_{\mathrm{c}}^{\bullet}(N;G) \simeq \varinjlim H_{\mathrm{c}}^{\bullet}(N_i;G).$$

But each N_i is compact, hence $H_{\mathrm{c}}^{\bullet}(N_i;G) \simeq H^{\bullet}(N_i;G)$. Note also that $C^{\bullet}(N;G) \simeq \varinjlim C^{\bullet}(N_i;G)$, in the category of cochain complexes. It remains to show that the functor $H^{\bullet}$ (of cochain complexes) commutes with taking colimits, which is a standard result in homological algebra. □

Remark 5.9. To compute the singular cohomology $H^{\bullet}(N;G)$, one may directly dualise $C_{\bullet}(N;G)$ by applying $\mathrm{Hom}_G(-, G)$. ◁

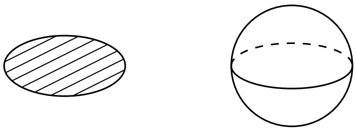
Remark 5.10. This gives a beautiful geometrical interpretation of Poincaré duality for a manifold M, which states that $H_{\bullet}(M;G)$ and $H_{\mathrm{c}}^{\bullet}(M;G)$ are dual to each other, provided that M is orientable or $2G = 0$. Namely, for a finite handlebody N, one defines in the obvious way the *dual handlebody* of N, which is diffeomorphic to N, and its λ-handles correspond to the $(n - \lambda)$-handles of N. Then the handle chain complex of N is isomorphic to the handle cochain complex of the dual of N. ◁

6 Application to 2-manifolds

This section gives an application of our theory of weak handlebodies. We will give a classification of 2-manifolds with finite topology. In particular, this will imply that $\mathbb{R}^2$ has a unique smooth structure.

We start from the simplest case.

Theorem 6.1. *Every simply connected boundaryless 2-manifold is diffeomorphic to either the open disk or the 2-sphere.*



This theorem is often proved as a consequence of the uniformisation theorem of Riemann surfaces, using the fact that every Riemannian 2-manifold can be given isothermal coordinates so that it becomes a Riemann surface. Here we give it a new and direct proof.

Proof. By (1.10) the manifold is diffeomorphic to a good 2-handlebody N. By (4.15) we may assume it is a good weak handlebody with only one 0-handle. Consider the handle chain complex with $\mathbb{Z}_2$ coefficients

$$\cdots \to 0 \to C_2(N;\mathbb{Z}_2) \xrightarrow{\partial_2} C_1(N;\mathbb{Z}_2) \xrightarrow{\partial_1} \mathbb{Z}_2.$$

Since $H_0(N;\mathbb{Z}_2) \simeq \mathbb{Z}_2$, we have $\partial_1 = 0$. Since N is simply connected, $H_1(N;\mathbb{Z}_2) = 0$. Thus ∂_2 is surjective. Note that the belt of a 1-handle is S^0, i.e. two points. Since ∂_2 is surjective, these two points must intersect with either (a) two different 2-handles, or (b) only one 2-handle in one point. Thus we have a graph G, with vertices corresponding to the 2-handles, and edges corresponding to 1-handles in case (a). For each 1-handle in case (b), we add an "external edge" that connects one vertex with "infinity". This can be interpreted as an infinite sequence of vertices and edges, so that G is rigorously a graph. This graph is locally finite, and does not have *loops* (i.e. edges whose endpoints are the same vertex).

We take a maximal tree in each connected component of G, and then apply (2.3) in the form of (4.13). The effect is that these trees are collapsed. By the same reason as above, the collapsed graph will not have loops. Thus the resulting graph will have no edges at all. This means that the resulting weak handlebody has no 1-handles. Therefore, if it has a 2-handle, then it is the sphere; if not, then it is the open disk. □

Corollary 6.2. $\mathbb{R}^2$ *has a unique smooth structure.* □

This method generalises to prove the following.

Theorem 6.3. *Every connected non-compact* 2*-manifold is diffeomorphic to a weak* $(2,1)$*-handlebody.*

Proof. Similarly, the manifold is diffeomorphic to a good weak 2-handlebody N with only one 0-handle. We construct something like a graph, denoted G, in the same way, except that edges (1-handles) need not have vertices (2-handles) as their endpoints. We ignore those edges with no endpoints for a while, and collapse maximal trees of connected components of the remaining part of G. The resulting thing should be a collection of vertices, each possibly with some loops attached to it, and some isolated edges.

We need to show that there are actually no vertices. Suppose the contrary. Then at some stage, a first 2-handle will be attached to a (non-weak) $(2,1)$-handlebody with only one 0-handle. Since every edge is a loop, it follows that whenever the image of the attaching map passes through a 1-handle, it passes through both sides of it. By an elementary argument in combinatorics, if it passes through a 1-handle, then it will pass through the whole boundary. This means that there will be no boundary after this attaching, and thus N must be compact, a contradiction. □

Together with the following well-known result, the preceding theorem will give a classification for all boundaryless 2-manifolds with finite topology.

The following result will also be proved in our handle-theoretic way.

Theorem 6.4 (Classification of closed surfaces). *Every connected closed 2-manifold is diffeomorphic to one of the following.*

(i) *The orientable surface* Σ_g := *connected sum of g tori, where* $g = 0, 1, \dots$ (where $\Sigma_0 := S^2$), *with Euler characteristic* $\chi(\Sigma_g) = 2 - 2g$.

(ii) *The non-orientable surface* Π_k := *connected sum of k projective planes, where* $k = 1, 2, \dots$, *with Euler characteristic* $\chi(\Pi_k) = 2 - k$.

Proof. Let N be a good handle decomposition. By (4.15), we assume that N has only one 0-handle. Taking the dual handlebody, we may assume N has only one 2-handle. Let k denote the number of 1-handles, and we prove by induction on k that the diffeomorphism type is decided by orientability and k.

If $k = 0$, then $N \simeq S^2$. If $k = 1$, an easy argument yields $N \simeq \mathbb{R}P^2$. Next we suppose $k \geq 2$. If we remove the 2-handle, we get a $(2, 1)$-handlebody with boundary S^1. If we remove one more 1-handle, one of the following happens.

(i) The boundary is two circles, and the surface is orientable. Thus the original surface is recovered by attaching a cylinder $S^1 \times I$ along these circles, and the orientability of the original surface decides how (regarding orientation) to attach this cylinder. If we instead attach two 2-handles along these circles, then the Euler characteristic will increase by 2. By the inductive hypothesis, the resulting surface S is decided by k. By (3.15) and (3.14), if we take two disjoint disks on S and remove their interiors, then the resulting surface S' is unique up to a diffeomorphism. Thus our original surface is decided by k.

(ii) The boundary is two circles, and the surface is non-orientable. This case is the same as (i) except that by (3.15), the two ways (regarding orientation) of attaching a cylinder result in the same surface.

(iii) The boundary is one circle. In this case the original surface must be non-orientable, and will be recovered by attaching a Möbius band along the circle. If we instead attach a 2-handle along the circle, then the Euler characteristic increases by 1. By a same argument as in (i), it remains to show that the connected sum $\Sigma_g \# \Pi_1$ is diffeomorphic to $\Pi_{2g} \# \Pi_1 \simeq \Pi_{2g+1}$. This follows from the observation that $\Pi_3 \simeq \Sigma_1 \# \Pi_1$. □

Now we may apply our results to non-compact manifolds. We say a manifold has *finite topology*, if its (say $\mathbb{R}$-coefficient) homology groups are finite dimensional.

Theorem 6.5 (Classification of 2-manifolds). *Every connected boundaryless 2-manifold with finite topology is diffeomorphic to either one of the closed surfaces* Σ_g *and* Π_k, *or one of them with a finite number of points removed.*

Proof. The compact case follows from the preceding theorem. For the non-compact case, by (6.3), the manifold must be a weak $(2, 1)$-handlebody with one

0-handle. By the finiteness assumption, it must have finitely many 1-handles. Thus it is the interior of a finite $(2,1)$-handlebody, whose boundary is a compact 1-manifold, i.e. a finite number of circles. After attaching 2-handles along these circles, it would become a closed surface, whence the result follows. □

The remaining case to a complete classification of 2-manifolds is that of weak $(2,1)$-handlebodies with one 0-handle and infinitely many 1-handles. However, classifying them is very difficult, and even the open subsets of $\mathbb{R}^2$ can not easily be classified. On the other hand, we can obtain partial results. For example, using the results in this paper, one can show that after multiplying by $\mathbb{R}$, all the orientable ones will become diffeomorphic 3-dimensional manifolds.

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