Homology of Configuration Spaces

Bu Chenjing

ABSTRACT

We propose a method of computing the (co)homology of configuration spaces of manifolds via a spectral sequence, and we describe this spectral sequence explicitly for elliptic curves.

Contents

Definition 0.1. Let X be a topological space, and let n be a non-negative integer. The *configuration space* of n points in X is the space

$$\mathrm{Conf}_n(X) = \{x_1, \dots, x_n \in X \mid x_i \neq x_j \ \forall i \neq j\},$$

equipped with the topology as an open subspace of X^n.

We aim to describe the (co)homology of $\mathrm{Conf}_n(X)$ when X is a manifold.

Notation 0.2. Throughout this paper, when we mention the (co)homology of a space, we always assume that a coefficient field $\Bbbk$ with characteristic zero is chosen, and we write $H_\bullet(X)$ for $H_\bullet(X; \Bbbk)$, etc.

1 Configuration spaces of euclidean spaces

The homology of $\mathrm{Conf}_n(\mathbb{R}^d)$ is well understood, and here we give a brief summary of the results. For more details, the reader is referred to, for example, [Sin06].

Definition 1.1. Let $n \geq 1$ be an integer. An *n-tree* is a binary tree with n leaves. For example,  and  are the only possible 3-trees. For an n-tree T, denote

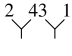
$$|T| = n - 1,$$

which is the number of internal vertices of T. These internal vertices are canonically ordered, with the ordering defined by traversing the tree in the order 'left–parent–right'. $\lhd$

Definition 1.2. Let $n \geq 0$ be an integer. An *n-forest* is a sequence of trees with n leaves in total, with the leaves labelled with $1, \dots, n$, without repeating. For a forest $F = (T_1, \dots, T_m)$, let

$$|F| = \sum_{i=1}^{m} |T_i| = n - m.$$

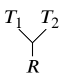
The $|F|$ internal vertices are ordered in the same way as in Definition 1.1, with the vertices of the first tree coming first, and then the second tree, and so on. $\lhd$

For example, $F =$ 

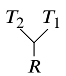
is a 4-forest, with $|F| = 2$.

Theorem 1.3. *Let $d \geq 2$ be an integer. Then the homology $H_\bullet(\mathrm{Conf}_n(\mathbb{R}^d))$ is the quotient of the free module generated by all n-forests by the following relations:*

- (anti-symmetry) *For any trees T_1, T_2, one has*



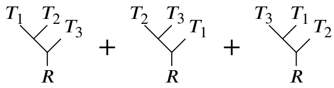
$$= -(-1)^{(d-1)(|T_1|+1)(|T_2|+1)}$$



,

where R can be the root, or can be attached to any leaf of a tree.

- (Jacobi identity) *For any trees* T_1, T_2, T_3, *one has*



$$= 0,$$

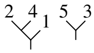
where R *can be the root, or can be attached to any leaf of a tree.*
- (orientation) *For any* n*-forest* $F = (T_1, \dots, T_m)$ *and any* $\sigma \in \mathfrak{S}_m$, *one has*

$$(T_{\sigma(1)}, \dots, T_{\sigma(m)}) = (-1)^{(d-1)\sigma'} (T_1, \dots, T_m),$$

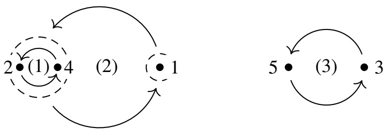
where $\sigma' \in \mathfrak{S}_{|F|}$ *is the induced permutation of the internal vertices of* F.

The homology class associated to a forest F *has homological degree* $(d-1)|F|$.

The generators of $H_\bullet(\mathrm{Conf}_n(\mathbb{R}^d))$ can be explicitly written down as 'orbital systems'. For example, the homology class

$F =$  $\in H_{3(d-1)}(\mathrm{Conf}_5(\mathbb{R}^d))$

is defined by the map $(S^{d-1})^3 \to \mathrm{Conf}_5(\mathbb{R}^d)$ depicted as follows, with $d = 2$.



The ordering of the factors S^{d-1} is shown as numbers with parentheses, and is determined by the ordering of internal vertices (Definition 1.2).

In fact, tree diagrams are closely related to Lie brackets. The previous theorem readily implies the following.

Corollary 1.4. *The space*

$$\mathrm{Tree}_n^{\pm} = (\text{linear span of } n\text{-trees})/\sim,$$

where $\sim$ *denotes anti-symmetry and the Jacobi identity, and* $\pm$ *is equal to* $(-1)^{d-1}$, *is isomorphic to the space*

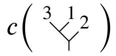
$$\mathrm{Lie}_n^{\pm} = \text{linear span of ways to take Lie brackets of } n \text{ elements},$$

where the n *elements are even* $(+)$ *or odd* $(-)$, *and each of the* n *elements is required to be used exactly once. Let*

$$c : \mathrm{Tree}_n^{\pm} \simeq \mathrm{Lie}_n^{\pm}$$

denote this isomorphism. $\square$

For example,



$$= [[3,1],2] = 312 \mp 132 - 231 \pm 213.$$

A richer version of Corollary 1.4 is stated as follows, which we do not attempt to prove here.

Corollary 1.5. *The homology of the little disks operad E_n is isomorphic to the Poisson operad P_n.*

Next, let us describe the cohomology ring $H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d))$.

Definition 1.6. Let $n \geq 0$ be an integer. By an *n-graph*, we mean a directed graph with n vertices, which are labelled $1, \dots, n$ without repeating, and with a specified ordering of its edges, such that it is acyclic in the sense that the induced undirected graph contains neither cycles nor multi-edges. For an n-graph Γ, denote

$$|\Gamma| = \#\mathrm{edges}(\Gamma). \qquad \triangleleft$$

Theorem 1.7. *Let $d \geq 2$ be an integer. Then the cohomology $H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d))$ is the quotient of the free module generated by all n-graphs by the following relations:*

- (anti-symmetry)

$$\begin{tikzcd}[sep=small] & j \\ i \arrow[ur] \end{tikzcd} \quad = \quad (-1)^d \begin{tikzcd}[sep=small] & j \arrow[dl] \\ i \end{tikzcd},$$

  where i, j may be connected to other edges as well.

- (Arnold identity)

$$\begin{tikzcd}[sep=small] & j \arrow[dr] & \\ i \arrow[ur] & & k \end{tikzcd} + \begin{tikzcd}[sep=small] & j \arrow[dr] & \\ i & & k \arrow[ll] \end{tikzcd} + \begin{tikzcd}[sep=small] & j & \\ i \arrow[ur] & & k \arrow[ll] \end{tikzcd} = 0,$$

  where i, j, k may be connected to other edges as well.

- (orientation) *If Γ' is obtained from Γ by reordering the edges using a permutation $\sigma \in \mathfrak{S}_{|\Gamma|}$, then*

$$\Gamma' = (-1)^{(d-1)\sigma}\Gamma.$$

The cohomology class associated to a graph Γ has cohomological degree $(d-1)|\Gamma|$. *The cup product is given by taking the disjoint union of edges* (if this does not make a valid n-graph, then the product is zero).

The elements $\begin{tikzcd}[sep=small] & j \\ i \arrow[ur] \end{tikzcd}$ form a set of generators of the cohomology ring. These generators can be expressed explicitly as the pullback of the fundamental class of S^{d-1} along the map

$$\begin{aligned} \alpha_{ij} : \mathrm{Conf}_n(\mathbb{R}^d) &\to S^{d-1}, \\ (x_1, \dots, x_n) &\mapsto \frac{x_i - x_j}{|x_i - x_j|}. \end{aligned}$$

Finally, we describe the cohomology–homology pairing in terms of forests and graphs.

Theorem 1.8. *Let F be an n-forest, and let Γ be an n-graph. If there exists a bijection*

$$f\colon \{\text{edges of } \Gamma\} \simeq \{\text{internal vertices of } F\},$$

such that for each edge $e = (i \to j)$ of Γ, the leaves i and j appear in the left and right (resp. right and left) *branches of the vertex $f(e)$, respectively, then putting $\varepsilon(e) = 1$* (resp. $(-1)^d$), *we have*

$$\langle \Gamma, F \rangle = (-1)^{(d-1)\sigma} \prod_e \varepsilon(e),$$

where $\sigma \in \mathfrak{S}_{|F|}$ is the permutation which relates the ordering of edges of Γ to the ordering of internal vertices of F. If such f does not exist, then $\langle \Gamma, F \rangle = 0$.

Proof. This follows directly from the explicit expressions of the generators of the (co)homology groups in terms of forests and graphs. □

Note that if such f exists, then $f(e)$ is uniquely determined, as for any two leaves in a binary tree, there is a unique internal vertex whose left and right branches each contain one of the two given leaves.

2 Configuration spaces of manifolds

In order to compute the homology of configuration spaces of manifolds, we use handle theory to break down the manifold into simpler pieces.

Let us first briefly recall the basics of handle theory.

Definition 2.1. For a fixed dimension d, and for $0 \le r \le d$, an *r-handle* is the manifold with corners

$$h^r = D^r \times D^{d-r},$$

which is to be regarded as a thickened version of the r-disk. A d-dimensional *handlebody* is a sequence of manifolds (possibly with boundaries)

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots,$$

either finite or infinite, such that each X_i is obtained from X_{i-1} by attaching a handle:

$$X_i = X_{i-1} \cup_{\Phi_i} h^r,$$

where

$$\Phi_i\colon \partial D^r \times D^{d-r} \to \partial X_{i-1}$$

is a smooth map, called the *attaching map*. The topological space

$$X = \operatorname*{colim}_{i \to \infty} X_i$$

is the *total space* of the handlebody. We require *local finiteness*, which states that every point of X has a neighbourhood which only intersects with the images of finitely many attaching maps. This ensures that X has an induced smooth structure.

By abuse of language, we call X a d-dimensional handlebody. ◁

It is a fundamental result in handle theory that every smooth manifold has a *handle decomposition*, that is, the structure of a handlebody.

We will need our handlebodies to satisfy an extra technical condition, in order to simplify our arguments later.

Definition 2.2. A handlebody X is said to be *perfect*, if the image of each attaching map

$$\Phi_i : \partial D^r \times D^{d-r} \to \partial X_{i-1}$$

for an r-handle is entirely contained in the boundaries of previously attached $(r-1)$-handles, and for each $z \in \partial D^{d-r}$, the $(r-1)$-dimensional slice

$$\Phi_i(\partial D^r \times z) \subset X_{i-1}$$

is a finite union of slices of the form $D^{r-1} \times z' \subset h^{r-1}$, where h^{r-1} is a previously attached $(r-1)$-handle, and $z' \in \partial D^{d-r+1}$. ◁

In fact, every smooth manifold has the structure of a perfect handlebody, which may be constructed from a triangulation. We omit the proof of this fact, which we will not use anyway.

In the following, let X be a perfect handlebody.

Definition 2.3. Let

$$\mathrm{Conf}_n'(X) \subset \mathrm{Conf}_n(X)$$

be the subspace of $\mathrm{Conf}_n(X)$ consisting of the configurations $x = (x_1, \dots, x_n)$, such that for every handle $h^r = D^r \times D^{n-r} \subset X$, and any $z \in D^{n-r}$, the intersection $x \cap (D^r \times z)$ contains at most one point. ◁

For example, if $x \in \mathrm{Conf}_n'(X)$, then every top dimensional handle of X is allowed to contain at most one point of x.

Lemma 2.4. $\mathrm{Conf}_n'(X)$ *is homotopy equivalent to* $\mathrm{Conf}_n(X)$.

Proof. Let $\mathring{X}$ denote the interior of X. Then $\mathrm{Conf}_n(\mathring{X})$ is exactly the interior of the manifold $\mathrm{Conf}_n(X)$ (possibly with corners). Therefore, $\mathrm{Conf}_n(X)$ is homotopy equivalent to $\mathrm{Conf}_n(\mathring{X})$, and $\mathrm{Conf}_n'(X)$ is homotopy equivalent to $\mathrm{Conf}_n'(X) \cap \mathrm{Conf}_n(\mathring{X})$, which we abbreviate as $\mathrm{Conf}_n'(\mathring{X})$.

Let $U_r \subset \mathrm{Conf}_n(\mathring{X})$ be the subspace such that the defining criterion of $\mathrm{Conf}_n'(X)$ is satisfied for all handles of type $\geq r$. Thus, $U_{d+1} = \mathrm{Conf}_n(\mathring{X})$, where $d = \dim X$, and $U_0 = \mathrm{Conf}_n'(\mathring{X})$. It suffices to show that U_{r+1} is homotopy equivalent to U_r for $r = 0, 1, \dots, d$.

For each handle h^r of X, let Y be the vector field on h^r defined by

$$Y = \nabla(|a|^2) \quad \text{for} \quad (a, b) \in D^r \times D^{d-r},$$

where the gradient is taken in euclidean space. By choosing an appropriate smooth structure on X (so that when two r-handles intersect at the boundary of an $(r-1)$-handle, their vector fields Y coincide on their intersection), the vector fields Y

extend to a globally defined vector field Y on X. Using an appropriate Riemannian metric on X, the norm of Y is bounded and X is complete, so that the flow φ_t of Y is defined for all $t \in \mathbb{R}$.

We observe that for any $x \in U_{r+1}$, the flow φ_t eventually takes x into U_r. Precisely speaking, for any $x \in U_{r+1}$, there exists a unique $T \geq 0$, such that for any $t > 0$, $\varphi_t(x)$ is in U_r if and only if $t > T$. We define

$$\rho\colon U_{r+1} \to V_r, \quad x \mapsto \varphi_T(x),$$

where T depends on x, and V_r is the subset of U_{r+1} consisting of those configurations x such that for any r-handle $D^r \times D^{d-r} \subset X$, and any $z \in D^{d-r}$, the intersection $x \cap (\mathring{D}^r \times z)$ contains at most one point, where $\mathring{D}^r$ denotes the interior of D^r. Then ρ is a deformation retraction map, and hence, a homotopy equivalence.

Finally, we need to show that V_r is homotopy equivalent to U_r. This follows from the fact that V_r is a (topological) manifold with boundary, whose interior is U_r. This can be proved by inspecting the neighbourhood of every point in V_r, and it is crucial to consider $\mathring{X}$ instead of X. We omit the full argument. □

We observe that $\mathrm{Conf}'_n(X)$ has the structure of a *topologically enriched CW complex*, described as follows.

Definition 2.5. A *topologically enriched CW complex* consists of the following data:

- A topological space $X[i]$, called the *space of i-cells*, for $i = 0, 1, 2, \dots$;
- A sequence of topological spaces $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots$;
- For each i, an *attaching map* $\Phi_i\colon X[i] \times \partial D^i \to X_{i-1}$,

such that

- For each i, one has
$$X_i = X_{i-1} \cup_{\Phi_i} (X[i] \times D^i).$$
- For each $z \in X[i]$, the image
$$\Phi_i(z \times \partial D^i) \subset X_{i-1}$$
is contained in a finite union of previously attached cells, i.e., a finite union of subsets of the form $\Phi_{i'}(z' \times D^{i'})$, where $i' < i$ and $z' \in X[i']$.

The space
$$X = \operatorname*{colim}_{i\to\infty} X_i$$
is called the *total space* of the topologically enriched CW complex. ◁

For example, a perfect handlebody may be seen as a topologically enriched CW complex, with the space $X[i]$ being a disjoint union of disks of dimension $\dim X - i$. This can be seen as a special case of the following lemma, with $n = 1$.

Lemma 2.6. *The space $C = \mathrm{Conf}'_n(X)$ has the structure of a topologically enriched CW complex, with homotopy equivalences*

$$C[i] \simeq \coprod_{\substack{n_1+\cdots+n_m=n \\ n_1r_1+\cdots+n_mr_m=i}} \left(\prod_{j=1}^{m} \mathrm{Conf}_{n_j}(\mathbb{R}^{d-r_j}) \right)^{\amalg \binom{n}{n_1 \; \cdots \; n_m}},$$

where m is the number of handles of X, each n_j is a non-negative integer, r_j is the type of the j-th handle of X, and d is the dimension of X.

Proof. For each choice of $n_1, \ldots, n_m$ satisfying the criterion of the coproduct, and for each choice of $z_j \in \mathrm{Conf}_{n_j}(D^{d-r_j})$ for $j = 1, \ldots, m$, we define

$$e_{z_1,\ldots,z_m} : (D^{r_1})^{n_1} \times \cdots \times (D^{r_m})^{n_m} \to X^n$$

by putting n_j points on the j-th handle, according the coordinates given by z_j and $(D^{r_j})^{n_j}$. Let $Z_{n_1,\ldots,n_m}$ be the space of all $(z_1, \ldots, z_m)$ such that the image of $e_{z_1,\ldots,z_m}$ lies in $\mathrm{Conf}'_n(X)$ (i.e., the points do not overlap). Then for each $z \in Z_{n_1,\ldots,n_m}$, the map e_z gives rise to an i-cell of $\mathrm{Conf}'_n(X)$. It follows that $C[i]$ is the disjoint union of all the spaces $Z_{n_1,\ldots,n_m}$, each with $\binom{n}{n_1 \; \cdots \; n_m}$ copies.

We notice that

$$\prod_{j=1}^{m} \mathrm{Conf}_{n_j}(\mathring{D}^{d-r_j}) \subset Z_{n_1,\ldots,n_m} \subset \prod_{j=1}^{m} \mathrm{Conf}_{n_j}(D^{d-r_j}),$$

where $\mathring{D}^{d-r_j}$ denotes the interior of D^{d-r_j}. However, the right hand side is a manifold with corners whose interior is the left hand side, so that $Z_{n_1,\ldots,n_m}$ is homotopy equivalent to both spaces. □

The (co)homology of a topologically enriched CW complex may be computed using cellular (co)homology.

Lemma 2.7. *Let X be a topologically enriched CW complex, such that each $X[i]$ is homotopy equivalent to a CW complex. Let*

$$A_{p,q} = C_p^{\mathrm{sing}}(X[q])$$

be the p-th term of the singular chain complex of $X[q]$. Let

$$\partial : A_{p,q} \to A_{p-1,q}$$

be the singular boundary map, and let

$$\partial^{\mathrm{cell}} : A_{p,q} \to A_{p,q-1}$$

be the cellular boundary map, defined in the same way as with ordinary CW complexes. Then the total complex of the double complex $(A_{p,q}, \partial, \partial^{\mathrm{cell}})$ computes the homology of X:

$$H_\bullet(X) \simeq H_\bullet(\mathrm{Tot}_\bullet(A_{\bullet,\bullet})).$$

Proof. Let

$$X'[i] = |\operatorname{Sing} X[i]|$$

be the geometric realisation of the simplicial set of singular simplices of $X[i]$. Then the spaces $X'[i]$, together with the induced attaching maps, give rise to a topologically enriched CW complex, which we denote by X'. Then X' carries a naturally induced CW structure (as an ordinary CW complex).

We notice that $\mathrm{Tot}_\bullet(A_{\bullet,\bullet})$ is isomorphic to the cellular chain complex of X'. Moreover, we have weak homotopy equivalences $X'[i] \simeq X[i]$, induced by the counit map, which are homotopy equivalences due to Whitehead's theorem. By induction on i, using [MP12, Lemmas 2.1.3], we see that X_i and X'_i are homotopy equivalent. By [MP12, Lemmas 2.1.10], we conclude that X and X' are homotopy equivalent. □

Corollary 2.8. *Suppose that each $X[i]$ has the homotopy type of a CW complex. Then there exists a spectral sequence*

$$E^1_{p,q} = H_p(X[q]) \Rightarrow H_{p+q}(X). \qquad \square$$

Analogously, one may compute the cohomology of a topologically enriched CW complex using cellular cohomology.

Lemma 2.9. *Let X be a topologically enriched CW complex, such that each $X[i]$ is homotopy equivalent to a CW complex. Let*

$$A^{p,q} = C^p_{\mathrm{sing}}(X[q])$$

be the p-th term of the singular cochain complex of $X[q]$. Let

$$d : A^{p,q} \to A^{p+1,q}$$

be the singular coboundary map, and let

$$d_{\mathrm{cell}} : A^{p,q} \to A^{p,q+1}$$

be the cellular coboundary map. Then the total complex of the double complex $(A^{p,q}, d, d_{\mathrm{cell}})$ computes the cohomology of X:

$$H^\bullet(X) \simeq H^\bullet(\mathrm{Tot}^\bullet(A^{\bullet,\bullet})). \qquad \square$$

Corollary 2.10. *Suppose that each $X[i]$ has the homotopy type of a CW complex. Then there exists a spectral sequence*

$$E_1^{p,q} = H^p(X[q]) \Rightarrow H^{p+q}(X). \qquad \square$$

Putting these results together, we are able to compute the (co)homology of configuration spaces, using topologically enriched cellular (co)homology.

Theorem 2.11. *Let X be a perfect handlebody. Then there exist spectral sequences*

$$E^1_{p,q} = \bigoplus_{\substack{n_1+\cdots+n_m=n\\ n_1r_1+\cdots+n_mr_m=q\\ k_1+\cdots+k_m=p}} \left(\bigotimes_{j=1}^{m} H_{k_j}(\mathrm{Conf}_{n_j}(\mathbb{R}^{d-r_j})) \right)^{\oplus \binom{n}{n_1 \cdots n_m}} \Rightarrow H_{p+q}(\mathrm{Conf}_n(X)),$$

$$E_1^{p,q} = \bigoplus_{\substack{n_1+\cdots+n_m=n\\ n_1r_1+\cdots+n_mr_m=q\\ k_1+\cdots+k_m=p}} \left(\bigotimes_{j=1}^{m} H^{k_j}(\mathrm{Conf}_{n_j}(\mathbb{R}^{d-r_j})) \right)^{\oplus \binom{n}{n_1 \cdots n_m}} \Rightarrow H^{p+q}(\mathrm{Conf}_n(X)),$$

where m is the number of handles of X, each n_j is a non-negative integer, r_j is the type of the j-th handle of X, and d is the dimension of X. □

Since these spectral sequences come from double complexes, for any given X, it is possible to write down the differential maps explicitly, and find out the (co)homology of $\mathrm{Conf}_n(X)$. We shall do this for elliptic curves in the next section.

3 Configuration spaces of elliptic curves

Let E be an elliptic curve over $\mathbb{C}$, which is homeomorphic to a torus.

Lemma 3.1. *There is a homeomorphism*

$$\mathrm{Conf}_{n+1}(E) \simeq E \times \mathrm{Conf}_n(E^*),$$

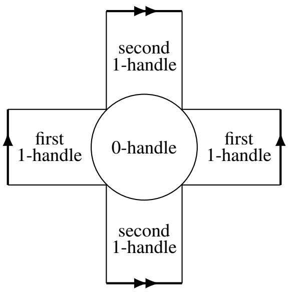
where $E^* = E \setminus \{*\}$.

Proof. This correspondence is given by

$$(x_1, \ldots, x_{n+1}) \mapsto \big(x_1, (x_2 - x_1, \ldots, x_{n+1} - x_1)\big),$$

where we have used the group structure on E, and we identified $E^* \simeq E \setminus \{0\}$. □

The space E^* homeomorphic to the interior of a handlebody X, which contains one 0-handle and two 1-handles, as shown below (edges with arrows are glued together).

$X =$ 

Note that $\mathrm{Conf}_n(E^*)$ is homotopy equivalent to $\mathrm{Conf}_n(X)$, since the latter space is a manifold (possibly with corners) whose interior is the former space. From now on, we aim to compute

$$H_\bullet(\mathrm{Conf}_n(E^*)) \simeq H_\bullet(\mathrm{Conf}_n(X)),$$

using this handlebody structure.

First, we introduce a notation for generators of $E^1_{p,q}$ in the spectral sequence of Theorem 2.11, for $X = E^*$.

Notation 3.2. We use the notation

$$(F, S_1, S_2),$$

where

- F is an n_0-forest with k_0 inner vertices (as a generator of $H_{k_0}(\mathrm{Conf}_{n_0}(\mathbb{R}^2))$);
- S_1 and S_2 are ordered sets of n_1 and n_2 elements, respectively (as generators of $H_0(\mathrm{Conf}_{n_j}(\mathbb{R}^1))$, for $j = 1, 2$),

with

$$n_0 + n_1 + n_2 = n,$$

and with the leaves of F and the elements of S_1, S_2 labelled $1, 2, \dots, n$ without repeating. This denotes the corresponding generator of $E^1_{p,q}$, with

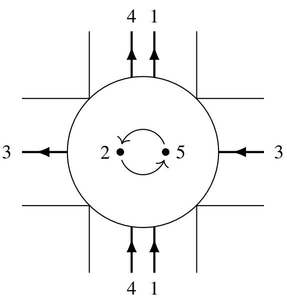
$$p = k_0, \quad q = n_1 + n_2. \qquad \triangleleft$$

Thus, if $(F, S_1, S_2) \in E^1_{p,q}$, then F is an $(n-q)$-forest with $|F| = p$, and $|S_1| + |S_2| = q$.

For example, the generator

$$\left({}^{2}\curlyvee^{5}\ ,\ 3\ ,\ 4\ 1\right) \in E^1_{1,3} = H_1\left(\mathrm{Conf}'_5(E^*)[3]\right)$$

may be depicted as follows:



Lemma 3.3. *In the spectral sequence $E^1_{p,q}$, one has*

$$d_1 = \partial^{\mathrm{cell}} = 0 : \ E^1_{p,q} \to E^1_{p,q-1} \,,$$

so that

$$E^2_{p,q} = E^1_{p,q} \,.$$

Proof. We have by definition

$$\begin{aligned} &\partial^{\mathrm{cell}}(F, S_1, S_2) \\ &= \sum_{i \in S_1 \cup S_2} \pm\big((F \cup i, S_1 \setminus \{i\}, S_2 \setminus \{i\}) - (F \cup i, S_1 \setminus \{i\}, S_2 \setminus \{i\})\big) \\ &= 0, \end{aligned}$$

as homology classes. □

Next, we compute d_2.

Lemma 3.4. *The map*

$$d_2 : \ E^2_{p,q} \to E^2_{p+1,q-2}$$

is given by

$$d_2(F, S_1, S_2) = \sum_{\substack{i_1 \in S_1 \\ i_2 \in S_2}} \pm\Big(\ F \cup \overset{i_1 \ i_2}{\curlyvee} \ , \ S_1 \setminus \{i_1\} \ , \ S_2 \setminus \{i_2\} \ \Big),$$

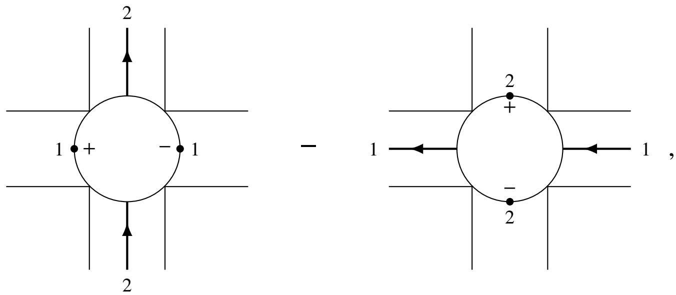
where the $\pm$ sign is determined by the Koszul sign rule. Explicitly, if i_k is the j_k-th element of S_k ($k = 1, 2$), then the sign is $(-1)^{|S_1| + j_1 + j_2 - 1}$.

Although this lemma is a special case of Theorem 3.11 below, we write down the construction explicitly to give the reader an idea of how the proof works.

Proof. For simplicity, we assume that $(F, S_1, S_2) = (\varnothing, 1, 2)$. The general case will follow from a similar argument.

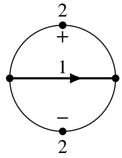
Denote $\gamma = (\varnothing, 1, 2)$. Then

$\partial^{\mathrm{cell}}\gamma \ =$

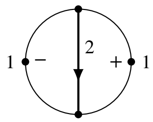


where $+$ and $-$ indicate the coefficients of the points. Thus,

$$\partial^{\mathrm{cell}}\partial^{-1}\partial^{\mathrm{cell}}\gamma \quad =$$



$+$



,

and this 1-cycle represents the points 1 and 2 orbiting each other for one counter-clockwise rotation. $\square$

Example 3.5. When $n = 2$, we find that

$$\left({}^{1}\!\curlyvee^{2}, \varnothing, \varnothing\right) = d_2(\varnothing, 1, 2) = d_2(\varnothing, 2, 1).$$

This means that the 1-cycle defined by two points orbiting each other is actually trivial. It also means that

$$(\varnothing, 1, 2) - (\varnothing, 2, 1)$$

survives to E^∞, and defines a 2-cycle. We may write down this 2-cycle explicitly: identifying $E \simeq \mathbb{R}^2/\mathbb{Z}^2$ and $E^* \simeq E \setminus \left(\frac{1}{2}, \frac{1}{2}\right)$, we define two maps

$$\begin{aligned} f, g \colon\ E \setminus (0,0) &\to \mathrm{Conf}_2(E^*), \\ f(x,y) &= \big((x,0),\ (0,y)\big), \\ g(x,y) &= \big((0,x),\ (y,0)\big). \end{aligned}$$

Putting them together (modifying them a little bit near $(0,0)$), we obtain a map from the connected sum of two tori to $\mathrm{Conf}_2(E^*)$, as a generator of $H_2(\mathrm{Conf}_2(E^*))$. $\lhd$

To describe the maps d_3, d_4, and so on, we need some preparations.

Construction 3.6. For each $p, q \geq 0$, we define a vector space

$$V^n_{p,q} = \bigoplus_{F,S} \Bbbk \cdot (F, S) \Big/ \sim,$$

where we sum over all pairs (F, S) such that

- F is an $(n-q)$-forest with $|F| = p$, and with its leaves labelled by $(n-q)$ distinct elements of $\{1, \ldots, n\}$;
- S is a permutation of the q numbers not used by F,

and $\sim$ denotes the relations of forests given in Theorem 1.3. $\lhd$

In particular, we notice that

$$\begin{aligned} V^n_{p,0} &\simeq H_p(\mathrm{Conf}_n(\mathbb{R}^2)), \\ V^n_{0,q} &\simeq \Bbbk[\mathfrak{S}_q], \end{aligned}$$

where $\mathfrak{S}_q$ denotes the permutation group.

Construction 3.7. We define a map

$$a\colon\ E^1_{p,q}\to V^n_{p,q}$$

by sending

$$\begin{aligned}(F,S_1,S_2)&\mapsto(F,[S_1,S_2])\\&=(F,S_1S_2)-(-1)^{|S_1||S_2|}(F,S_2S_1).\end{aligned}$$ ◁

Construction 3.8. For each $p\geq 0$ and $q\geq 1$, we define a subspace $W^n_{p,q}\subset V^n_{p,q}$, and a pairing

$$\langle -,-\rangle\colon\ H^{p+q-1}(\mathrm{Conf}_n(\mathbb{R}^2))\otimes W^n_{p,q}\to\Bbbk,$$

as follows.

Let Γ be an n-graph with $|\Gamma|=p+q-1$, and let $(F,S)\in V^n_{p,q}$ be a generator. The pairing $\langle\Gamma,(F,S)\rangle$ is only non-zero when, up to a reordering of the edges, Γ can be written as

$$\Gamma=\Gamma_1\cup\Gamma_2,$$

where $\cup$ denotes the union of edges, which coincides with the cup product, such that the edges of Γ_1 only connects vertices labelled with numbers used by F, and the edges of Γ_2 only connects vertices labelled with numbers used by S, with $|\Gamma_1|=p$ and $|\Gamma_2|=q-1$. In this case, we define

$$\langle\Gamma,(F,S)\rangle=\langle\Gamma_1,F\rangle\langle\Gamma_2,S\rangle,$$

where $\langle\Gamma_1,F\rangle$ is defined in Theorem 1.8 (with $d=2$), and

$$\langle\Gamma_2,S\rangle=\det(\varepsilon_{ij})_{1\leq i,j\leq q-1},$$

where ε_{ij} is given as follows: if the i-th edge of Γ_2 is $k_i\to l_i$, and the j-th element of S is s_j, then

$$\varepsilon_{ij}=\begin{cases}\delta_{s_jk_i}-\delta_{s_jl_i}, & \text{if } k_i \text{ precedes } l_i \text{ in } S,\\ 0, & \text{otherwise.}\end{cases}$$

Finally, let $W^n_{p,q}\subset V^n_{p,q}$ be the subspace on which this pairing is well-defined, that is, the relations satisfied by graphs have to be satisfied by this pairing.

Note that we could have defined (ε_{ij}) as a $(q-1)\times q$ matrix, but all its $(q-1)\times(q-1)$ minors are equal (up to a sign), as all the row sums of (ε_{ij}) are 0. Moreover, the determinant must be 0 or ±1, as can be proved by induction on q. ◁

Using this pairing, one may associate homology classes to elements of $W^n_{p,q}$.

Construction 3.9. For $p\geq 0$ and $q\geq 1$, let

$$b\colon\ W^n_{p,q}\to W^n_{p+q-1,0}\simeq H_{p+q-1}(\mathrm{Conf}_n(\mathbb{R}^2))$$

be the map induced by the pairing $\langle -,-\rangle$, so that for any $w\in W^n_{p,q}$ and any n-graph Γ with $|\Gamma|=p+q-1$, one has

$$\langle\Gamma,b(w)\rangle=\langle\Gamma,w\rangle.$$ ◁

The spirit of the map b is that it gives some sort of an 'inverse' to the map c defined in Corollary 1.4, as is shown in the following example.

Example 3.10. Let F be an $(n-q)$-forest with $|F| = p$, and let T be a q-tree, with their n-leaves labelled $1, \dots, n$. Then $(F, c(T))$ is in $W^n_{p,q}$, where c is the map defined in Corollary 1.4 (with $d = 2$), and

$$b(F, c(T)) = F \cup T.$$

To prove this, it suffices to show that

$$\langle \Gamma, T \rangle = \langle \Gamma, c(T) \rangle$$

for any graph Γ and any tree T, as one can verify as an exercise in linear algebra. $\lhd$

Theorem 3.11. *Let $r \geq 1$, and suppose that we have an element of $E^r_{p,q}$ of the form*

$$\gamma = \sum_\alpha c_\alpha \cdot (F, S^\alpha_1, S^\alpha_2),$$

where the forest F does not depend on α. Then

$$d_r(\gamma) = \sum_{S'_1, S'_2} \left(b \circ a \left(\sum_{\substack{S'_1 \subset S^\alpha_1 \\ S'_2 \subset S^\alpha_2}} (-1)^{\sigma_\alpha} c_\alpha \cdot (F,\ S^\alpha_1 \setminus S'_1,\ S^\alpha_2 \setminus S'_2) \right),\ S'_1,\ S'_2 \right),$$

where

- *The sum is taken over all sequences S'_1, S'_2 of distinct elements of $\{1, \dots, n\}$, such that $S'_1 \cap S'_2 = \varnothing$ and $|S'_1| + |S'_2| = n - r$.*
- *The notation $S'_k \subset S^\alpha_k$ denotes inclusion preserving order $(k = 1, 2)$.*
- *$\sigma_\alpha \in \mathfrak{S}_q$ is the permutation satisfying*

$$\sigma_\alpha(S^\alpha_1, S^\alpha_2) = (S^\alpha_1 \setminus S'_1,\ S^\alpha_2 \setminus S'_2,\ S'_1,\ S'_2).$$

- *That the expression $b \circ a(\cdots)$ is well-defined, i.e. the element $a(\cdots)$ lies in $W^n_{p,q}$ for each choice of S'_1 and S'_2, is a part of the statement of the theorem.*

We postpone the proof of the theorem to Appendix A.

Corollary 3.12. *Suppose that F is a forest, and L, R are two trees, such that they have n leaves in total, labelled $1, \dots, n$. Then the element*

$$\gamma = (F, c(L), c(R))$$

satisfies

$$\begin{aligned} d_1(\gamma) &= \cdots = d_{q-1}(\gamma) = 0, \\ d_q(\gamma) &= \left(F \cup {}^{L}\!\curlyvee^{R}, \varnothing, \varnothing\right), \end{aligned}$$

where $q = |L| + |R| + 2$.

Proof. For any tree T, and any non-empty sequence S' of distinct labels used by T, we have

$$c(T) \setminus S' = 0,$$

where we extend the map $(-) \setminus S'$ linearly (together with the appropriate $\pm$ sign), and we define $S \backslash S' = 0$ if $S' \not\subset S$. Applying this to $T = {}^{L}\!\curlyvee^{R}$ explains why $d_1(\gamma) = \cdots = d_{q-1}(\gamma) = 0$. The expression for $d_q(\gamma)$ comes from Example 3.10. $\square$

Corollary 3.13. *For any $p \geq 1$, one has*

$$E^{p+2}_{p,0} = 0.$$

Proof. The space $E^1_{p,0}$ is generated by the elements $(F, \varnothing, \varnothing)$, where F is a forest. Upon reordering the trees in F, we may assume that $F = F' \cup T$, where T is the tree in F such that it has the least leaves among the trees that have more than one leaf. Let $t = |T|$, and suppose that $T = {}^{L}\!\curlyvee^{R}$. Then

$$(F, \varnothing, \varnothing) = d_{t+1}(F', c(L), c(R)).$$

Therefore, the element $(F, \varnothing, \varnothing)$ is eliminated by d_{t+1}. Since t never exceeds $|F| = p$, it follows that d_{p+1} must be surjective, whence the corollary. $\square$

Corollary 3.14. *For any $n \geq 0$, we have*

$$\begin{aligned} h_1(\mathrm{Conf}_n(E^*)) &= 2n, \\ h_2(\mathrm{Conf}_n(E^*)) &= \frac{n(n-1)(2n+11)}{6}, \end{aligned}$$

so that

$$\begin{aligned} h_1(\mathrm{Conf}_n(E)) &= 2n, \\ h_2(\mathrm{Conf}_n(E)) &= \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Proof. The E^2 page is shown below, where each number in the grid represents the dimension of the corresponding vector space, and the number on an arrow represents the rank of that arrow.

$$\begin{tikzcd}
q & & & \\
& 4n(n-1)(n-2) \arrow[ddr, "\frac{2n(n-1)(n-2)}{3}"] & \cdots \arrow[ddr] & \\
& 3n(n-1) \arrow[ddr, "\frac{n(n-1)}{2}"'] & \cdots \arrow[ddr] & \\
& 2n & n(n-1)(n-2) & \cdots \\
& 1 & \frac{n(n-1)}{2} & \cdots \\
& & & p
\end{tikzcd}$$

The rank $n(n-1)/2$ follows from Corollary 3.13, and the rank $2n(n-1)(n-2)/3$ comes from

$$d_2(F, i, j\ k) = (F \cup {}^{i}\mathsf{Y}^{j}, \varnothing, k) - (F \cup {}^{i}\mathsf{Y}^{k}, \varnothing, j),$$

where $|F| = 0$, so that each choice of i, j, k gives two generators of $E^3_{1,1}$:

$$\begin{aligned}
(\cdots {}^{i}\mathsf{Y}^{j}, \varnothing, k) &= (\cdots {}^{j}\mathsf{Y}^{k}, \varnothing, i) = (\cdots {}^{k}\mathsf{Y}^{i}, \varnothing, j), \\
(\cdots {}^{i}\mathsf{Y}^{j}, k, \varnothing) &= (\cdots {}^{j}\mathsf{Y}^{k}, i, \varnothing) = (\cdots {}^{k}\mathsf{Y}^{i}, j, \varnothing).
\end{aligned}$$

By Corollary 3.13 again, $E^4_{2,0} = 0$, so that the E^∞ page looks like this:

$$\begin{array}{c|ccc}
q & & & \\
 & \cdots & & \\
 & \frac{5n(n-1)}{2} & \cdots & \\
 & 2n & \frac{n(n-1)(n-2)}{3} & \cdots \\
 & 1 & 0 & 0 \\
\hline
 & & & p
\end{array}$$

This implies the formulas for $h_p(\mathrm{Conf}_n(E^*))$ for $p = 1, 2$. For $h_p(\mathrm{Conf}_n(E))$, one simply applies the Künneth formula. $\square$

Corollary 3.15. *The homology groups of* $\mathrm{Conf}_n(E^*)$ *and* $\mathrm{Conf}_n(E)$*, for small values of n, are given as follows.*

n	$h_\bullet(\mathrm{Conf}_n(E^*))$	$h_\bullet(\mathrm{Conf}_{n+1}(E))$
0	1	1, 2, 1
1	1, 2	1, 4, 5, 2
2	1, 4, 5	1, 6, 14, 14, 5
3	1, 6, 17, 18	1, 8, 30, 58, 53, 18
4	1, 8, 38, 88, 81	1, 10, 55, 172, 295, 250, 81
…	…	…

Proof. Theorem 3.11 covers all the needed maps d_r in order to compute these numbers, and the rank of the maps d_r may be computed using a computer. $\square$

Likewise, for the cohomological spectral sequence, the generators of $E_1^{p,q}$ can be written as

$$(\Gamma, S_1, S_2),$$

where Γ is an $(n-q)$-graph, and S_1, S_2 are as above. We formulate a conjecture on the mixed Hodge structure on the cohomology of $\mathrm{Conf}_n(E^*)$.

Conjecture 3.16. *The mixed Hodge structure on $H^k(\mathrm{Conf}_n(E^*);\mathbb{C})$ may be described as follows. The weight filtration on $E_1^{p,q}$ is given by*

$$W_l E_1^{p,q} = \bigoplus_{2|\Gamma|+|S_1|+|S_2|\le l} \mathbb{C}\cdot(\Gamma, S_1, S_2),$$

and the Hodge filtration is given by

$$F^a E_1^{p,q} = \bigoplus_{|\Gamma|+|S_1|\ge a} \mathbb{C}\cdot\big((\Gamma, S_1, S_2) + \mathrm{i}(\Gamma, S_2, S_1)\big).$$

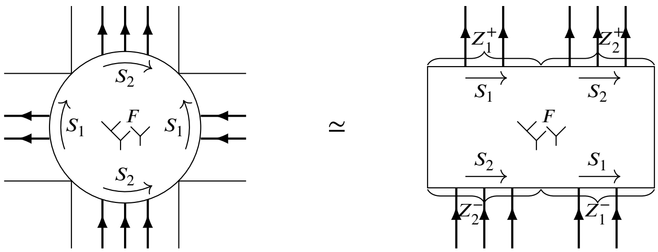
The induced weight and Hodge filtrations on $E_\infty^{p,q}$ determine the mixed Hodge structure on $H^k(\mathrm{Conf}_n(E^*);\mathbb{C})$.

Appendix A Proof of Theorem 3.11

The goal of this appendix is to study the map

$$d_r = (\partial^{\mathrm{cell}} \circ \partial^{-1})^{r-1} \circ \partial^{\mathrm{cell}}$$

in the spectral sequence $E_{p,q}^r$. For this purpose, we make the following identification, where (F, S_1, S_2) denotes a generator of $E_{p,q}^1$.



Let P denote this rectangle, and let $Z_1^\pm, Z_2^\pm \subset P$ denote the four open intervals as marked in the figure. Let Z_1 denote the topological space $Z_1^+ \simeq Z_1^-$, and similarly for Z_2. Let $Z^\pm \subset P$ denote the upper and lower horizontal edges (as closed intervals), and let $Z \subset P$ denote their union.

Let

$$C'_{\bullet,q} = \bigoplus_{n_1+n_2=q} \Big(C_\bullet^{\mathrm{rect}}(\mathrm{Conf}_{n-q}(P)) \otimes C_\bullet^{\mathrm{rect}}(\mathrm{Conf}_{n_1}(Z_1)) \otimes C_\bullet^{\mathrm{rect}}(\mathrm{Conf}_{n_2}(Z_2)) \Big)^{\oplus \binom{n}{n-q,\, n_1,\, n_2}},$$

where we take the tensor product of chain complexes, and

$$C_\bullet^{\mathrm{rect}}(\mathrm{Conf}_n(X)) \subset C_\bullet^{\mathrm{sing}}(X)^{\otimes n} \xrightarrow{\sim} C_\bullet^{\mathrm{sing}}(X^n)$$

denotes the subcomplex spanned by the elements $\sigma_1 \otimes \cdots \otimes \sigma_n$, where each σ_i is a simplex in X, such that the image of $\sigma_1 \times \cdots \times \sigma_n$ in X^n is contained in $\mathrm{Conf}_n(X)$. We interpret the multiplicity $\binom{n}{n-q, n_1, n_2}$ as taking $n-q$ points from $\{1, \dots, n\}$ to place on P, n_1 points to place on Z_1, and n_2 points on Z_2.

Let

$$C_{p,q} \subset C'_{p,q}$$

be the subspace spanned by those generators (f, z_1, z_2), such that both $f \times z_1^+ \times z_2^+$ and $f \times z_1^- \times z_2^-$, as singular chains in P^n, are supported in $\mathrm{Conf}_n(P)$, where $z_k^\pm$ denotes the push-forward of z_k along the inclusion map $Z_k^\pm \hookrightarrow P$. Then there are boundary maps

$$\partial : C_{p,q} \to C_{p-1,q}, \quad \partial^{\mathrm{cell}} : C_{p,q} \to C_{p,q-1},$$

which make $C_{\bullet,\bullet}$ into a double complex, where ∂^{cell} is defined on the generators by

$$\partial^{\mathrm{cell}}(f, z_1, z_2) = \sum_{j=1}^{q} (-1)^{j-1} \big(f \times (i_*^+ p_j - i_*^- p_j),\ z_1 \setminus p_j,\ z_2 \setminus p_j \big),$$

where p_j denotes the j-th point of the sequence $z_1 z_2$, and $i^\pm : Z_1^\pm \cup Z_2^\pm \hookrightarrow P$ denotes the inclusion map. The notations $i_*^\pm p_j$ and $z_k \setminus p_j$ ($k = 1, 2$) are only suggestive — precisely, one considers the singular chains $f \times z_1^+ \times z_2^+$ and $f \times z_1^- \times z_2^-$ of $\mathrm{Conf}_n(P)$, and then, one regards them as elements of $C_{p,q-1}$ by moving p_j to the first component.

Note that the double complex $C_{\bullet,\bullet}$ perfectly describes what happens to the 0-handle in the actual spectral sequence $E^r_{p,q}$. In fact, it is not difficult to see the following.

Lemma A.1. *The spectral sequence of the double complex $C_{\bullet,\bullet}$ is isomorphic to the spectral sequence $E^\bullet_{\bullet,\bullet}$ from the first page onward.* □

Therefore, from now on in this appendix, we will regard $E^\bullet_{\bullet,\bullet}$ as the spectral sequence of $C_{\bullet,\bullet}$. Note also that we have a natural forgetful map

$$g : C_{p,q} \hookrightarrow C_p^{\mathrm{sing}}(\mathrm{Conf}_{n-q}(P))^{\oplus \binom{n}{q}}.$$

Definition A.2. Let X be a topological space, and let α be a singular (resp. Borel–Moore) n-chain of X. The *canonical decomposition* of α is the unique way to write α as a finite (resp. countable) sum

$$\alpha = \sum_i a_i A_i,$$

where $a_i \in \Bbbk$ are distinct non-zero elements, and $A_i \subset \mathrm{Sing}\, X$ are simplicial subsets, such that each (non-degenerate) n-simplex of $\mathrm{Sing}\, X$ is contained in at most one A_i. ◁

Definition A.3. Let X be a topological manifold of dimension n. Let α be a Borel–Moore k-chain of X, and let β be a singular $(n-k)$-chain of X. Let $\alpha = \sum_i a_i A_i$ and $\beta = \sum_j b_j B_j$ be their canonical decompositions, and denote $A = \coprod_i A_i$ and $B = \coprod_j B_j$.

We say that α and β *intersect transversely*, if there only exists finitely many pairs $(x_A, x_B) \in A \times B$, such that the images of x_A, x_B in X coincide, and all such pairs (x_A, x_B) lie in $(A \setminus \partial A) \times (B \setminus \partial B)$.

In this case, the *intersection* $\alpha \cap \beta$ is a well-defined 0-cycle of X. ◁

For each k, let

$$Y \subset \mathrm{Conf}_k(P)$$

denote the set of configurations where all the points have the same y-coordinate. Then Y can be regarded as a submanifold of $\mathrm{Conf}_k(P)$ of codimension $k-1$.

The orientation of Y is given as follows. For any $x \in Y$, the isomorphism

$$T_x \mathrm{Conf}_k(P) \ \simeq \ T_x Y \oplus \mathbb{R} \cdot (p_1 \text{ moves down}) \oplus \cdots \oplus \mathbb{R} \cdot (p_{k-1} \text{ moves down})$$

preserves orientation, where p_i is the i-th point from the left to the right, according to the configuration x.

Definition A.4. An element $\gamma \in C_{k-1,n-k}$ is said to be a *good cycle*, if $\partial\gamma = 0$, and each of the $\binom{n}{k}$ components of $g(\gamma)$ intersects transversely with Y, with the intersection (as a 0-cycle) supported in $Y \cap \mathrm{Conf}_k(Z) \subset \mathrm{Conf}_k(P)$.

A good cycle $\gamma \in C_{k-1,n-k}$ is said to be *simple*, if precisely one of the $\binom{n}{k}$ components of $g(\gamma)$ is non-zero. ◁

For a simple good cycle $\gamma \in C_{k-1,n-k}$, the homology class of $g(\gamma)$ is determined by its intersection with Y, which lies in $\mathrm{Conf}_k(Z)$.

Lemma A.5. *Let γ be a simple good cycle, and suppose that*

$$Y \cap g(\gamma) = \sum_{i=1}^{m} c_i x_i,$$

with $c_i \in \Bbbk$ and $x_i \in \mathrm{Conf}_k(P)$. Then for any $(k-1)$-graph Γ, if we regard Γ as an element of $H^{k-1}(\mathrm{Conf}_k(P))$, then

$$\langle \Gamma, g(\gamma) \rangle = \sum_{i=1}^{m} c_i \, \langle \Gamma, S_i \rangle,$$

where S_i is the element of $\mathfrak{S}_k$ given by the horizontal arrangement of points at x_i, and $\langle \Gamma, S_i \rangle$ is defined in Construction 3.8.

Proof. We observe that the inclusion $\mathrm{Conf}_k(P) \hookrightarrow \mathrm{Conf}_k(\mathbb{R}^2)$ is a homotopy equivalence, and we write Γ for the corresponding cohomology class of $\mathrm{Conf}_k(\mathbb{R}^2)$. Using the duality between cohomology classes and Borel–Moore homology classes, we represent Γ as the Borel–Moore $(k+1)$-cycle Y_Γ, defined as follows. Let $i_r \to j_r$ be the r-th edge of Γ. Then Y_Γ is the union of the connected components of Y (whose definition is extended from $\mathrm{Conf}_k(P)$ to $\mathrm{Conf}_k(\mathbb{R}^2)$) in which the i_r-th point is on the left of the j_r-th point for all r. The orientation of Y_Γ is given so that at any $x \in Y_\Gamma$, the isomorphism

$$\begin{aligned} &T_x\mathrm{Conf}_k(\mathbb{R}^2) \simeq \\ &\qquad T_xY_\Gamma \oplus \mathbb{R}\cdot(i_1\text{-th point moves down and } j_1\text{-th point moves up}) \oplus \\ &\qquad\qquad \cdots \oplus \mathbb{R}\cdot(i_{k-1}\text{-th point moves down and } j_{k-1}\text{-th point moves up}) \end{aligned}$$

preserves orientation. We thus have

$$\langle \Gamma, g(\gamma)\rangle = [Y_\Gamma \cap g(\gamma)] \in H_0(\mathrm{Conf}_k(\mathbb{R}^2)).$$

At each x_i, either $x_i \notin Y_\Gamma$, in which case $\langle \Gamma, S_i\rangle = 0$, or $x_i \in Y_\Gamma$ and the orientation of Y_Γ differs from that of Y by a sign, which is equal to $\langle \Gamma, S_i\rangle$ by the definition of the latter. □

Lemma A.6. *Let $\gamma \in C_{k-1,n-k}$ be a good cycle, and suppose that there exists $\beta \in C_{k,n-k}$ such that $\gamma = \partial\beta$. Then β can be chosen so that $\partial^{\mathrm{cell}}\beta$ is a good cycle with*

$$g(Y \cap \partial^{\mathrm{cell}}\beta) = \frac{1}{k+1}\partial^{\mathrm{cell}}(Y \cap g(\gamma))|_Y,$$

where $(-)|_Y$ denotes removing from an element of $C_0^{\mathrm{sing}}(\mathrm{Conf}_{k+1}(Z))$ the points that are not in Y.

Proof. Let $g(\beta) = \sum_i a_i B_i$ be the canonical decomposition, where each B_i is a simplicial subset of $\mathrm{Sing}(\mathrm{Conf}_k(P))$, and let

$$A_i = \partial B_i \cap \big(\text{support of } g(\gamma) \text{ in } \mathrm{Sing}(\mathrm{Conf}_k(P))\big).$$

Then $g(\gamma)$ is represented by $\sum_i a_i A_i$.

We modify the $(k+1)$-cycles B_i, so that they intersect transversely with Y. Denote $B = \bigcup_i B_i \subset \mathrm{Sing}(\mathrm{Conf}_k(P))$, and let $A = \bigcup_i A_i \subset B$. The natural map $b\colon B \to \mathrm{Conf}_k(P)$ is homotopic to a map $b'\colon B \to \mathrm{Conf}_k(P)$ rel A, such that $b'(B \setminus A)$ is contained in $\mathrm{Conf}_k(P \setminus Z)$. This can be shown by induction on the dimension of simplices. From now on, we replace β with the singular chain represented by b'.

Since $\partial^{\mathrm{cell}}\beta$ is supported in the subset of $\mathrm{Conf}_{k+1}(P)$ with at least one of the $(k+1)$ points in Z, it follows that $\partial^{\mathrm{cell}}\beta$ can only intersect with Y in $\mathrm{Conf}_{k+1}(Z)$. At these intersection points, $g(\beta)$ meets with $\mathrm{Conf}_k(Z)$, so $g(\gamma)$ must meet with $\mathrm{Conf}_k(Z)$ by the construction of b'.

Let $x \in \mathrm{Conf}_{k+1}(P)$ be a point where $g(\partial^{\mathrm{cell}}\beta)$ meets Y. Let $v\colon \mathrm{Conf}_{k+1}(P) \to (\mathbb{R}_{\geq 0})^{k+1}$ be the projection map which takes the (euclidean) distance from each point to Z, so that $v(x) = 0$. Let $\beta_1, \dots, \beta_{k+1}$ be the $(k+1)$ of the $\binom{n}{k}$ components of $\partial^{\mathrm{cell}}\beta$ which concern k of the $(k+1)$ points in question, with β_i missing the

i-th point, and let $\partial' \beta_i$ be the component of $\partial^{\text{cell}} \beta_i$ that concerns these $(k+1)$ points. Denote $\beta' = \sum_i \beta_i$. Then $v_*(\partial' \beta')$ is a k-cycle supported in $\partial((\mathbb{R}_{\geq 0})^{k+1})$, with each $v_*(\partial' \beta_i)$ supported in a face $(\mathbb{R}_{\geq 0})^k$. Since $v_*(\partial' \beta')$ is a cycle, each $\partial v_*(\partial' \beta_i)$ is supported in $\partial((\mathbb{R}_{\geq 0})^k)$. If we consider its local (at x) intersection in $\partial((\mathbb{R}_{\geq 0})^k)$ with the point 0, we obtain a number t_i. Replacing the point 0 by a point in $\{y_i = y_j = 0\}$ near 0, where y_i and y_j are the coordinates for $\mathbb{R}^{k+1}$, we see that $t_i = t_j$, since $\sum_i \partial v_*(\partial' \beta_i) = 0$. Hence, all the numbers t_i are equal, and we call this number t. It follows that the intersection of the cycle $v_*(\partial' \beta')$, in $\partial((\mathbb{R}_{\geq 0})^{k+1})$, with the point 0, is also equal to t. Finally, we notice that t is the coefficient of x in $Y \cap g(\partial^{\text{cell}} \beta)$ and in $Y \cap g(\gamma)$. We have a coefficient $1/(k+1)$, because every term on the left corresponds to $(k+1)$ terms on the right. □

Proof of Theorem 3.11. For the sake of simplicity, we assume that $F = \emptyset$; the general case only involves minimal modification of the argument. We thus have $(p, q) = (0, n)$.

Let $\gamma \in C_{0,n}$ be a representative of the homology class which is denoted by γ in the theorem. We prove by induction on r that if we make appropriate choices when taking ∂^{-1} of an element, then the cycle

$$d_r(\gamma) = (\partial^{\text{cell}} \circ \partial^{-1})^{r-1} \circ \partial^{\text{cell}}(\gamma) \in C_{r-1,n-r}$$

is a good cycle, with

$$Y \cap g_{S_1', S_2'}(d_r(\gamma)) = \sum_{\substack{S_1' \subset S_1^\alpha \\ S_2' \subset S_2^\alpha}} (-1)^{\sigma_\alpha} \, c_\alpha \cdot \left(x^+_{\alpha, S_1', S_2'} - (-1)^{|S_1^\alpha \setminus S_1'| \, |S_2^\alpha \setminus S_2'|} x^-_{\alpha, S_1', S_2'} \right),$$

where the notations are as in the theorem; $g_{S_1', S_2'}$ denotes taking the components where the second and third parts are in the connected components given by S_1' and S_2', and then apply g; the points $x^{\pm}_{\alpha, S_1', S_2'} \in \mathrm{Conf}_r(Z^{\pm})$ are given by taking the points in $S_1^\alpha \setminus S_1'$ and $S_2^\alpha \setminus S_2'$ from our chosen representative γ. In fact, the case $r = 1$ is clear, and the induction step follows from Lemma A.6 in a straightforward way, as long as one keeps track of the signs.

Combining this with Lemma A.5, we arrive at the formula given in the theorem. □

References

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