Stacks and combinatorics in enumerative geometry

Chenjing Bu 卜辰璟
University of Oxford

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Overview

Enumerative geometry is the study of moduli spaces:

$$\text{moduli space } M \overset{\text{extract}}{\Longrightarrow} \text{enumerative invariants}$$

(numbers, spaces, categories, ...)

These include

- intersection pairings on $\mathrm{H}^\bullet(M)$;
- the Euler characteristic $\chi(M) = \sum_i (-1)^i \dim \mathrm{H}^i(M)$;
- the cohomology $\mathrm{H}^\bullet(M)$ (vector space, Hodge structure, etc.);
- the category $\mathrm{Coh}(M)$; ...

Overview

Obstacles

- M can be non-compact.
  - Construct a compactification; problem-specific.
- M can be singular.
  - Use virtual (i.e. derived) geometry instead of classical geometry.
- Points in M have automorphisms $\implies$ M is an (Artin) stack.
  - Need stack theory: Techniques only available in linear case.
  - **This talk:** The general case.
  - (Problem does not appear in Gromov–Witten theory.)

Overview

Obstacles

For example, consider the stack $M = */\mathbb{C}^\times$ ($\approx \mathbb{CP}^\infty$).

- Its Euler characteristic is $\chi(M) = 1/0 = \infty$.
- Its cohomology $\mathrm{H}^\bullet(M) \simeq \mathbb{Q}[x]$ is ∞-dimensional.
- It is difficult to make sense of intersection pairings on stacks.

Not easy to extract finite invariants in any of these flavours.

But some techniques are available when M parametrize objects in a linear category.

Overview

The linear case

- M = moduli of coherent sheaves on a variety X:
  - $\dim X = 1$:
    - intersection pairings: Jeffrey–Kiem–Kirwan–Woolf 2006, B 2023, B–Kiem 2025
    - cohomology: Mozgovoy–Reineke 2015
  - $\dim X = 2$: Donaldson invariants; Vafa–Witten invariants
  - X: Calabi–Yau 3-fold: Donaldson–Thomas invariants
  - X: Calabi–Yau 4-fold: DT4 invariants (not yet well-developed)
- M = moduli of representations of a quiver Q:
  - Donaldson–Thomas invariants

Overview

Goal

- Interpret these invariants as intrinsic to the moduli stack, without reference to a linear category.
- Generalize these invariants to general stacks, such as
  - Moduli of G-bundles or G-Higgs bundles, for any reductive group G.
  - GIT quotient stacks X/G.
  - ...
- Motivations from physics; Langlands duality; non-abelian Hodge theory; geometric representation theory; ...

Overview

Idea

The key tool is the component lattice of a stack.

- It generalizes root data and Weyl groups in representation theory.
- It describes the combinatorial structure of parabolic induction.
- It encodes the axiomatics of Hall algebras in linear moduli problems, and generalizes them to arbitrary stacks.

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Groups

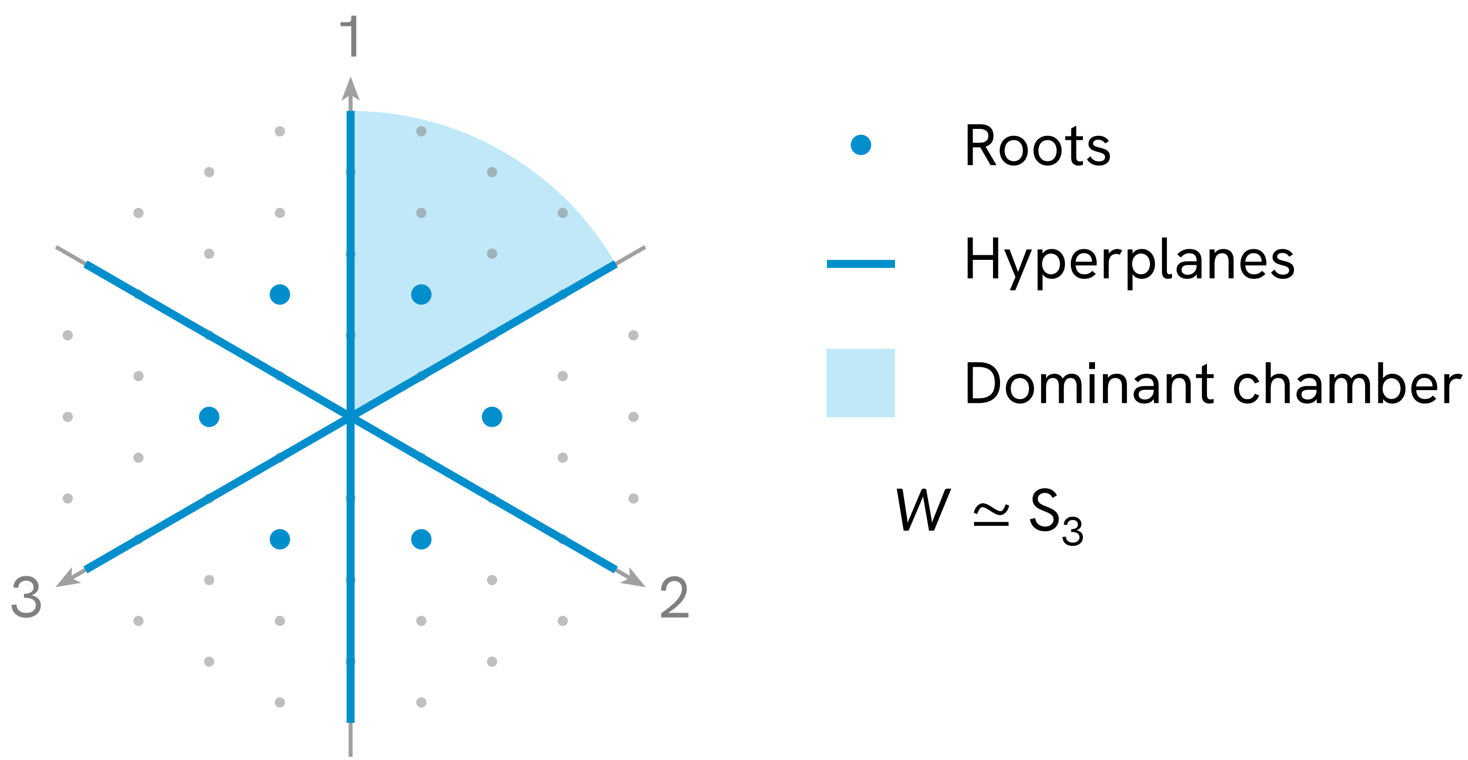
Set-up

- G: reductive group over $\mathbb{C}$, with Lie algebra $\mathfrak{g}$.
  Think: $G = \mathrm{GL}_n(\mathbb{C})$.
- $T \simeq (\mathbb{C}^\times)^n \subset G$: maximal torus.
  Think: diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$.
- $\Lambda^T = \mathrm{Hom}(T, \mathbb{C}^\times) \simeq \mathbb{Z}^n$: the character lattice.
  - There are roots $\Phi \subset \Lambda^T$, weights of T acting on $\mathfrak{g}$.
- $\Lambda_T = \mathrm{Hom}(\mathbb{C}^\times, T) \simeq \mathbb{Z}^n$: the cocharacter lattice.
  - Roots define hyperplanes in Λ_T, giving a hyperplane arrangement.
- W: Weyl group, acts on Λ_T and Λ^T via reflections along roots.

Groups

Example

$G = SL_3$. We have $\dim \Lambda^T = \dim \Lambda_T = 2$. There are 6 roots.



Groups

Definition (B–Halpern-Leistner–Ibáñez Núñez–Kinjo, 2025 preprint)

The component lattice of the stack $*/G$ is

$$\mathrm{CL}(*/G) := \Lambda_T/W,$$

the cocharacter lattice divided by the Weyl group.

Remarks

- As a set, it agrees with the dominant chamber Λ_T^+.
- It carries extra structure of a formal lattice over $\mathbb{Z}$. This is like taking a 'quotient stack' Λ_T/W.

Groups

Facts

- Cocharacters of G are given by

$$\frac{\{\lambda \colon \mathbb{C}^\times \to G\}}{\text{conjugation}} \simeq \Lambda_T / W.$$

- Representations of G split into irreducible representations, and

$$\left\{ \begin{matrix} \text{irreducible} \\ \text{representations} \end{matrix} \right\} \simeq \Lambda^T / W.$$

Dominant weight $\chi \in \Lambda^T \longleftrightarrow$ highest weight representation V_χ.

Groups

Facts

- The cohomology of the classifying space of G is

$$\mathrm{H}^\bullet(*/G;\mathbb{Q}) \simeq \mathbb{Q}[x_1,\dots,x_t]^W \simeq \left\{ \begin{matrix} \text{polynomial functions} \\ \Lambda_T/W \to \mathbb{Q} \end{matrix} \right\},$$

where $x_1,\dots,x_t$ is a set of coordinates on Λ_T.

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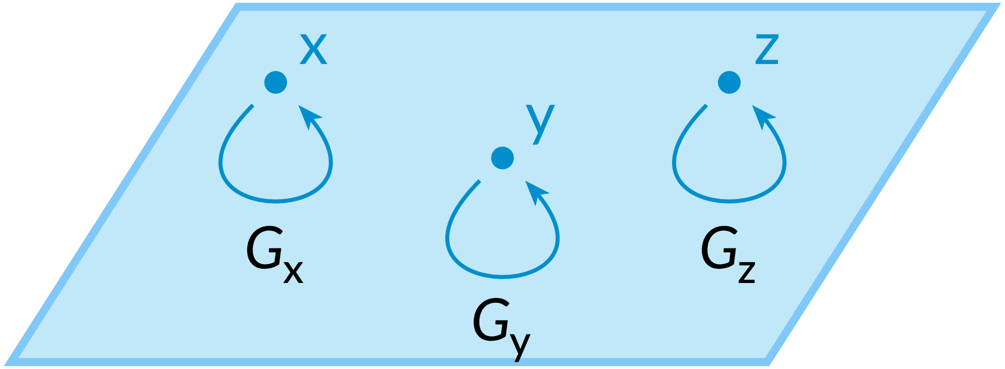
Stacks

Stacks

- A stack is roughly the same as a Lie groupoid, that is a groupoid whose objects and morphisms form manifolds or schemes.



- Many categories in algebraic geometry, such as Coh(X) or Rep(Q), can be upgraded to stacks, called moduli stacks.

Stacks

Stacks

- Cohomology of stacks generalize equivariant cohomology of schemes:

$$\mathrm{H}^\bullet(X/G;\mathbb{Q}) \simeq \mathrm{H}^\bullet_G(X;\mathbb{Q}) \, ,$$

where X is a scheme, and X/G is the quotient stack.

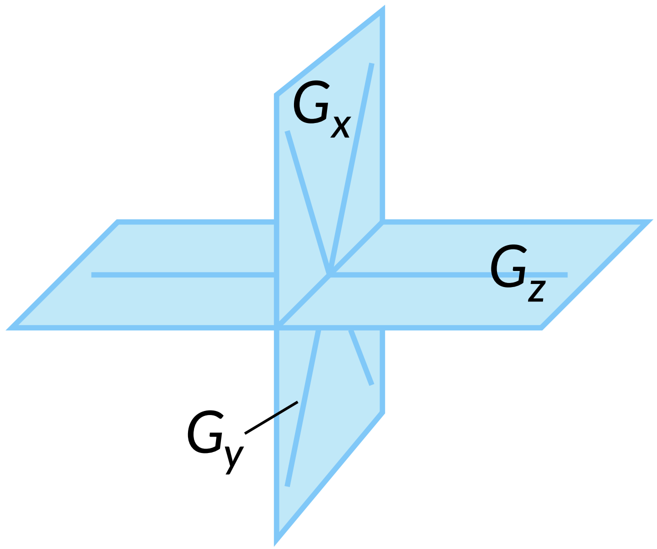
- Coherent sheaves on stacks generalize representations of algebraic groups:

$$\mathrm{Coh}(*/G) \simeq \mathrm{Rep}(G) \, .$$

Stacks

The component lattice

- The component lattice $\mathrm{CL}(\boldsymbol{X})$ of a stack $\boldsymbol{X}$ is similar to the lattice Λ_T/W for a group.
- For each point $x \in \boldsymbol{X}$, consider its automorphism group G_x and its lattice Λ_T/W, then glue them together.



Stacks

Definition (B–Halpern-Leistner–Ibáñez Núñez–Kinjo, 2025 preprint)

For a stack $\boldsymbol{X}$ over $\mathbb{C}$, define its component lattice

$$\mathrm{CL}(\boldsymbol{X}) = \pi_0\Big(\, \mathrm{Map}(*/\mathbb{C}^\times, \boldsymbol{X}) \,\Big)\,,$$

where

- π_0 means taking the set of connected components.
- $\mathrm{Map}(-,-)$ is the mapping stack.

Stacks

$$\mathrm{CL}(\boldsymbol{X}) = \pi_0\Big(\, \mathrm{Map}(*/\mathbb{C}^\times,\ \boldsymbol{X})\, \Big)$$

Remarks

- $\mathrm{CL}(\boldsymbol{X})$ is the set of equivalence classes of cocharacters in $\boldsymbol{X}$:

$$\left\{ \begin{matrix} \text{maps} \\ */\mathbb{C}^\times \to \boldsymbol{X} \end{matrix} \right\} \simeq \left\{ (x, \lambda) \;\middle|\; \begin{matrix} x \in \boldsymbol{X}\,, \\ \lambda \colon \mathbb{C}^\times \to G_x \end{matrix} \right\},$$

  where G_x is the automorphism group of x.

- $\mathrm{Map}(*/\mathbb{C}^\times, \boldsymbol{X}) = \mathrm{Grad}(\boldsymbol{X})$ is the stack of graded points of $\boldsymbol{X}$:
  - If $\boldsymbol{X}$ parametrizes objects in an abelian category $\mathcal{A}$, then $\mathrm{Grad}(\boldsymbol{X})$ parametrizes $\mathbb{Z}$-graded objects in $\mathcal{A}$.

Stacks

Combinatorial structure

The component lattice $\mathrm{CL}(\boldsymbol{X})$ has the structure of a formal lattice.

- A formal lattice is any functor

$$L\colon \{\text{ finite rank free } \mathbb{Z}\text{-modules }\}^{\mathrm{op}} \longrightarrow \{\text{ sets }\}\,.$$

- For example, all lattices $\mathbb{Z}^n$ are formal lattices.
- All limits and colimits of $\mathbb{Z}^n$ are formal lattices. For example,

$$\mathbb{Z}^n \sqcup \mathbb{Z}^m\,, \qquad \mathbb{Z}^n \cup_{\{0\}} \mathbb{Z}^m\,, \qquad \mathbb{Z}^n/G$$

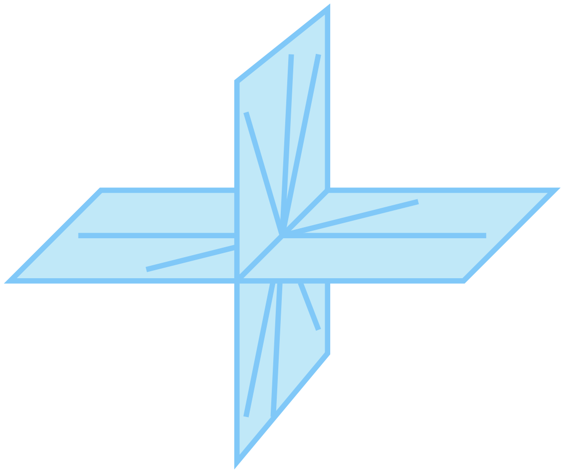
  are formal lattices, where a group G acts on $\mathbb{Z}^n$ linearly.

Stacks

Combinatorial structure

Moreover, $\mathrm{CL}(\boldsymbol{X})$ carries a natural wall-and-chamber structure:

- Automorphism groups G_x act on the tangent complex $\mathbb{T}_{\boldsymbol{X}}|_x$.
- The weights of this action define dual hyperplanes on $\mathrm{CL}(\boldsymbol{X})$.



- This is the key combinatorial data for enumerative geometry.

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Applications

Cohomological DT theory

For a symmetric quiver Q, and the moduli stack $\boldsymbol{X}$ of $\mathrm{Rep}(Q)$ over $\mathbb{C}$, Kontsevich–Soibelman (2011) conjectured that

$$\mathrm{H}^{\bullet}(\boldsymbol{X}; \mathbb{Q}) \simeq \mathrm{Sym}\Bigg(\bigoplus_{d\text{: dim vector}} \mathrm{BPS}_d \otimes \mathbb{Q}[t] \Bigg),$$

i.e., the cohomological Hall algebra (CoHA) is freely generated by certain vector spaces BPS_d, which categorify DT invariants.

- Efimov (2012) proved their conjecture.
- Meinhardt–Reineke (2019) related BPS_d to intersection cohomology.
- Davison–Meinhardt (2020) generalized it to quivers with potentials.

Applications

Theorem (B–Davison–Ibáñez Núñez–Kinjo–Pădurariu, 2025 preprint; Hennecart–Kinjo, 2025 preprint)

For a smooth symmetric stack $\boldsymbol{X}$ over $\mathbb{C}$ with a good moduli space X (and other mild assumptions), we have

$$H^\bullet(\boldsymbol{X};\mathbb{Q}) \simeq \bigoplus_{\alpha\colon \Lambda \to \mathrm{CL}(\boldsymbol{X})} \big(\mathrm{BPS}_\alpha \otimes \mathbb{Q}[t_1, \ldots, t_{\dim \Lambda}]\big)^{\mathrm{Aut}(\alpha)},$$

where

- α runs through walls in $\mathrm{CL}(\boldsymbol{X})$.
- $\mathrm{BPS}_\alpha \simeq \mathrm{IH}^\bullet(X_\alpha)$ is the (finite-dimensional) intersection cohomology of the good moduli space of a stack $\boldsymbol{X}_\alpha$ (when the stable locus is non-empty, or zero otherwise).

Applications

$$\mathrm{H}^\bullet(\boldsymbol{X};\mathbb{Q}) \simeq \bigoplus_{\alpha\colon\Lambda\to\mathrm{CL}(\boldsymbol{X})} \big(\mathrm{BPS}_\alpha \otimes \mathbb{Q}[t_1,\dots,t_{\dim\Lambda}]\big)^{\mathrm{Aut}(\alpha)}$$

Similar statements hold for Borel–Moore homology for symplectic stacks and critical cohomology for (−1)-shifted symplectic stacks.

Remarks

- $\boldsymbol{X}_\alpha$ is roughly a torus fixed locus in $\boldsymbol{X}$.
- $\mathrm{Aut}(\alpha)$ plays a similar role to Weyl groups.
- This recovers known statements in the quiver case.

Applications

Categorical DT theory

For a reductive group G, recall the orthogonal decomposition

$$\mathrm{Rep}(G) = \bigoplus_{\chi \in \Lambda^T/W} \langle V_\chi \rangle ,$$

where

- V_χ is the irreducible representation with higher weight χ.
- $\langle V_\chi \rangle \subset \mathrm{Rep}(G)$ is the abelian subcategory generated by V_χ.

Applications

Theorem (B–Pădurariu–Toda, in progress)

For a smooth symmetric stack $\boldsymbol{X}$ over $\mathbb{C}$ with a good moduli space X (and other mild assumptions), we have a semiorthogonal decomposition

$$\mathrm{D^bCoh}(\boldsymbol{X}) \simeq \langle \, \mathrm{W}_\lambda \mid \lambda \in \mathrm{CL}(\boldsymbol{X}) \otimes \mathbb{Q} \, \rangle \, ,$$

where

- $\mathrm{W}_\lambda \subset \mathrm{D^bCoh}(\boldsymbol{X}_\lambda)$ is a window subcategory.
- A quadratic form on $\mathrm{CL}(\boldsymbol{X})$ is needed to convert λ to a character.

We expect similar decompositions for DT categories of symplectic and (-1)-shifted symplectic stacks.

Applications

Geometric representation theory

For a reductive group G, Lusztig's generalized Springer theory gives an orthogonal decomposition

$$\mathrm{Perv}(\mathcal{N}/G) \simeq \bigoplus_{(L,C)} \mathrm{Rep}(W_{G,L}) ,$$

where

- $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone.
- $L \subset G$ is a Levi subgroup, and C is a cuspidal local system on a nilpotent orbit of L.
- $W_{G,L}$ is the relative Weyl group.

Applications

Mackey formula (B–Hennecart, in progress)

For a smooth stack $\boldsymbol{X}$ over $\mathbb{C}$ with a good moduli space, $\lambda, \mu \in \mathrm{CL}(\boldsymbol{X})$, we have roughly

$$\mathrm{Res}_\mu \circ \mathrm{Ind}_\lambda \sim \bigoplus_w \mathrm{Ind}_{\boldsymbol{X}_\mu, \lambda} \circ \mathrm{Res}_{\boldsymbol{X}_\lambda, w\mu} \colon \ \mathrm{D}^{\mathrm{b}}_{\mathrm{con}}(\boldsymbol{X}_\lambda) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{con}}(\boldsymbol{X}_\mu)\,,$$

where

- $\sim$ means roughly that l.h.s. has a filtration by the r.h.s.
- For Springer theory, take $\boldsymbol{X} = \mathfrak{g}/G$, so $\boldsymbol{X}_\lambda = \mathfrak{l}_\lambda / L_\lambda$ (Levi subgroup).

This may give decompositions for perverse sheaves on general stacks.

Thank you!