A theory of type B/C/D enumerative invariants

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Contents

Overview

Enumerative invariants.

- They are numbers that count geometric objects.
- More formally, they count points in moduli spaces, or compute intersection pairings in moduli spaces.
- This is equivalent to defining and computing (virtual) fundamental classes of moduli spaces.
- They can be tricky to define when moduli spaces are singular and fail to have fundamental classes.

Overview

Examples. Counting objects in an abelian category:

- Counting quiver representations:
  - Donaldson–Thomas (DT) invariants; Joyce's invariants.
- Counting coherent sheaves:
  - On curves: fundamental classes computed by Witten and Jeffrey–Kirwan.
  - On surfaces: Donaldson invariants; Seiberg–Witten invariants; Joyce's invariants.
  - On threefolds: DT invariants.

Overview

All the above are type A invariants, for structure groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$, or $\mathrm{U}(n)$ and $\mathrm{SU}(n)$.

Goal.

- Generalize this to type B/C/D invariants, including structure groups $G = \mathrm{O}(n)$ or $\mathrm{Sp}(2n)$.

Examples.

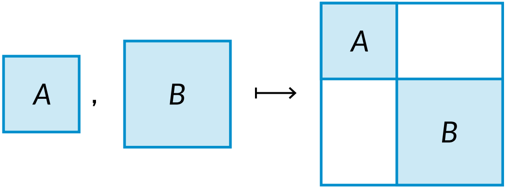
- Counting principal G-bundles.
- Counting G-quiver representations.
- Counting G-Higgs bundles.

Overview

Algebraic structures.

- In type A, invariants interact with algebraic structures such as Hall algebras and Joyce vertex algebras.
- Roughly, these structures come from the operation

$$\mathrm{GL}(n) \times \mathrm{GL}(m) \xrightarrow{\oplus} \mathrm{GL}(n+m)$$



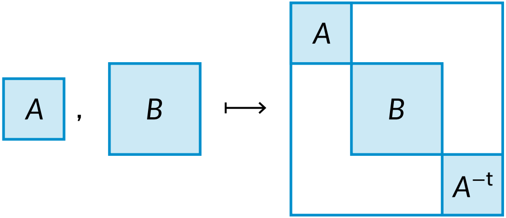
Overview

Algebraic structures.

- In type B/C/D, we get modules for these algebras. Roughly,

$$\mathrm{GL}(n) \times \mathrm{O}(m) \xrightarrow{\oplus^{\mathrm{sd}}} \mathrm{O}(2n+m)$$

$$\mathrm{GL}(n) \times \mathrm{Sp}(2m) \xrightarrow{\oplus^{\mathrm{sd}}} \mathrm{Sp}(2n+2m)$$

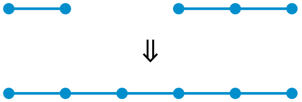


Overview

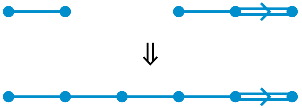
Algebraic structures.

These can also be seen in terms of Dynkin diagrams:

- In type A:



- In type B/C/D:



Overview

Algebraic structures.

(Co)homology theory	Type A $\mathrm{GL}(n)$	Type B/C/D $\mathrm{O}(n)$, $\mathrm{Sp}(2n)$
motivic ring $\mathrm{SF}(\mathcal{M})$	motivic Hall algebra (associative algebra)	module
	$\Rightarrow$ Lie algebra	twisted module
cohomology $H^\bullet(\mathcal{M})$	cohomological Hall algebra (CoHA)	module (CoHM)
homology $H_\bullet(\mathcal{M})$	Joyce vertex algebra	twisted module
	$\Rightarrow$ Lie algebra	twisted module

Contents

Self-dual categories

Definition. A self-dual category consists of

- A category $\mathcal{A}$.
- An equivalence $(-)^\vee : \mathcal{A} \xrightarrow{\sim} \mathcal{A}^{\mathrm{op}}$.
- A natural isomorphism $\eta : (-)^{\vee\vee} \xRightarrow{\sim} (-)$,

satisfying compatibility conditions.

Example. $\mathcal{A} = \mathrm{Vect}(X)$, vector bundles on a space X.

- $(-)^\vee$ takes the dual vector bundle.
- η can be ± 1.

Self-dual categories

Definition. A self-dual object (E, ϕ) in a self-dual category $\mathcal{A}$ consists of

- An object $E \in \mathcal{A}$.
- An isomorphism $\phi \colon E \xrightarrow{\sim} E^\vee$ with $\phi = \phi^\vee$.

Example. For vector bundles $E \in \mathrm{Vect}(X)$,

- $\phi \colon E \xrightarrow{\sim} E^\vee \iff$ non-degenerate bilinear form on E.
- $\phi = \phi^\vee \iff$ the form is
  - symmetric if $\eta = 1$,
  - alternating if $\eta = -1$.
- The self-dual objects are
  - orthogonal bundles if $\eta = 1$,
  - symplectic bundles if $\eta = -1$.

Self-dual categories

Module structure.

- An additive category $\mathcal{A}$ looks like an algebra:

$$\begin{aligned} \oplus \colon \mathcal{A} \times \mathcal{A} &\longrightarrow \mathcal{A} \, , \\ (E, F) &\longmapsto E \oplus F \, . \end{aligned}$$

- If $\mathcal{A}$ is self-dual, then the groupoid $\mathcal{A}^{\mathrm{sd}}$ of self-dual objects is like a module for $\mathcal{A}$:

$$\begin{aligned} \oplus^{\mathrm{sd}} \colon \mathcal{A}^{\simeq} \times \mathcal{A}^{\mathrm{sd}} &\longrightarrow \mathcal{A}^{\mathrm{sd}} \, , \\ \big(E, (F, \phi)\big) &\longmapsto (E \oplus F \oplus E^{\vee}, \tilde{\phi}) \, , \end{aligned}$$

  with $\tilde{\phi}$ given by ϕ and the natural pairing between E and $E^{\vee}$.

Self-dual categories

Moduli spaces.

- Assume $\mathcal{M}$ is a moduli space for $\mathcal{A}$.
- The self-dual structure on $\mathcal{A}$ induces a $\mathbb{Z}_2$-action

$$(-)^\vee : \mathcal{M} \xrightarrow{\sim} \mathcal{M} .$$

- The homotopy fixed points of this action

$$\mathcal{M}^{\mathrm{sd}} = \mathcal{M}^{\mathbb{Z}_2}$$

  is a moduli space for $\mathcal{A}^{\mathrm{sd}}$.

Self-dual categories

Example.

- There is a $\mathbb{Z}_2$-action

$$(-)^\vee : \mathrm{BGL}(n) \xrightarrow{\sim} \mathrm{BGL}(n),$$

  given by the map $A \mapsto A^{-t}$ on $\mathrm{GL}(n)$, or taking the dual vector space. The homotopy fixed points are

$$\mathrm{BGL}(n)^{\mathbb{Z}_2} \simeq \mathrm{BO}(n) \quad \text{or} \quad \mathrm{BSp}(n),$$

  depending on $\eta = \pm 1$. (We set $\mathrm{BSp}(\mathrm{odd}) = \varnothing$.)

- For a space X, can apply $\mathrm{Map}(X, -)$ to the above.

  $\Longrightarrow$ O/Sp bundles on X are $\mathbb{Z}_2$-fixed vector bundles.

Self-dual categories

Self-dual stability conditions.

A stability condition

$$\tau : \{\text{non-zero } E \in \mathcal{A}\} \longrightarrow \mathbb{R}$$

(or any totally ordered set)

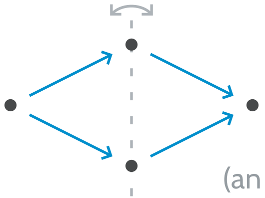
is self-dual if $\tau(E) = -\tau(E^\vee)$ for all non-zero $E \in \mathcal{A}$.

- Self-dual objects always have slope 0, i.e. $\tau(E) = 0$.
- Can define τ-(semi)stable self-dual objects.

Contents

Self-dual quivers

Definition (Derksen–Weyman, Young).



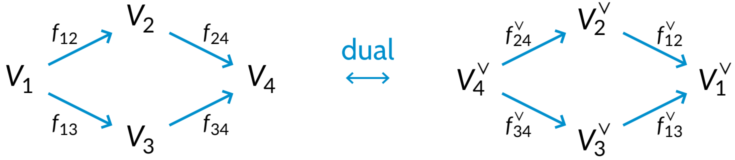
(an example)

A self-dual quiver is a quiver Q together with

- An involution $(-)^\vee : Q \xrightarrow{\sim} Q^{\mathrm{op}}$.
- A sign $u_i \in \{\pm 1\}$ for each vertex i.
- A sign $v_j \in \{\pm 1\}$ for each edge j.

Self-dual quivers

Self-dual structure on $\mathcal{A}$ = {representations of Q}:



A self-dual object in $\mathcal{A}$ satisfies, in this example:

- $V_1 \simeq V_4^\vee$.
- $V_2 \simeq V_2^\vee$ and $V_3 \simeq V_3^\vee$ via orthogonal or symplectic structures, depending on the signs u_2, u_3.
- $f_{12} = v_{24}\, f_{24}^\vee$, etc.

Contents

Hall algebras and modules

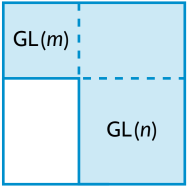
Extensions.

- In an exact category $\mathcal{A}$, a short exact sequence

$$0 \to E_1 \longrightarrow E \longrightarrow E_2 \to 0$$

  is an extension of E_2 by E_1 .

- Think of as principal bundles for the parabolic subgroup



$$\subset \quad \mathrm{GL}(m+n) \, .$$

Hall algebras and modules

Motivic Hall algebra.

- $\mathcal{M}^{\mathrm{ex}}$: moduli of extensions, or short exact sequences in $\mathcal{A}$.
- We have maps

$$\begin{array}{ccc} & \mathcal{M}^{\mathrm{ex}} & \\ {}^{(\pi_1,\pi_2)}\swarrow & & \searrow^{\pi} \\ \mathcal{M}\times\mathcal{M} & & \mathcal{M}\,, \end{array} \qquad \begin{array}{ccc} & (0 \to E_1 \to E \to E_2 \to 0) & \\ \swarrow & & \searrow \\ (E_1, E_2) & & E \end{array}$$

- Associative product $*$ on the motivic ring $\mathrm{SF}(\mathcal{M})$ given by

$$* \;=\; \pi_* \circ (\pi_1, \pi_2)^* \,.$$

This is the motivic Hall algebra.

Hall algebras and modules

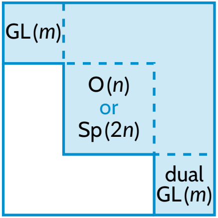
Self-dual extensions.

- A self-dual extension of $(F, \psi) \in \mathcal{A}^{\mathrm{sd}}$ by $E_1 \in \mathcal{A}$ is a filtration

$$0 \simeq E_0 \overset{E_1}{\hookrightarrow} E_1 \overset{F}{\hookrightarrow} E_2 \overset{E_1^\vee}{\hookrightarrow} E_3 \simeq E$$

of $(E, \phi) \in \mathcal{A}^{\mathrm{sd}}$, with quotients $E_1, F, E_1^\vee$ compatible with ϕ.

- Think of as principal bundles for the parabolic subgroup



$\subset$ $\mathrm{O}(2m+n)$ or $\mathrm{Sp}(2m+2n)$.

Hall algebras and modules

Motivic Hall module.

- $\mathcal{M}^{\mathrm{sdex}}$: moduli of self-dual extensions.
- We have maps

$$\begin{array}{ccc} & \mathcal{M}^{\mathrm{sdex}} & \\ {\scriptstyle (\pi_1,\pi_2)} \swarrow & & \searrow {\scriptstyle \pi} \\ \mathcal{M} \times \mathcal{M}^{\mathrm{sd}} & & \mathcal{M}^{\mathrm{sd}} \end{array}, \qquad \begin{array}{ccc} & (0 \overset{E_1}{\hookrightarrow} E_1 \overset{F}{\hookrightarrow} E_2 \overset{E_1^\vee}{\hookrightarrow} E_3 = E) & \\ \swarrow & & \searrow \\ (E_1, F) & & E \end{array}$$

- The motivic Hall algebra $\mathrm{SF}(\mathcal{M})$ acts on $\mathrm{SF}(\mathcal{M}^{\mathrm{sd}})$ by

$$\diamond = \pi_* \circ (\pi_1, \pi_2)^* .$$

This defines the motivic Hall module $\mathrm{SF}(\mathcal{M}^{\mathrm{sd}})$.

Hall algebras and modules

Lie algebras and twisted modules.

- $\mathrm{SF}(\mathcal{M})$ is also a Lie algebra:

$$[a, b] = a * b - b * a \, .$$

  It has an involution $(-)^\vee$ satisfying

$$[a^\vee, b^\vee] = [b, a]^\vee \, .$$

- It acts on $\mathrm{SF}(\mathcal{M}^{\mathrm{sd}})$ by

$$a \heartsuit m = a \diamond m - a^\vee \diamond m \, .$$

  This is a twisted module:

$$a \heartsuit b \heartsuit m - b \heartsuit a \heartsuit m = [a, b] \heartsuit m - [a^\vee, b] \heartsuit m \, .$$

Contents

Motivic invariants

- Motivic enumerative invariants are elements

$$\epsilon_\alpha(\tau) \in \mathrm{SF}(\mathcal{M}), \qquad \text{(Joyce} \sim 2007\text{)}$$
$$\epsilon^{\mathrm{sd}}_\theta(\tau) \in \mathrm{SF}(\mathcal{M}^{\mathrm{sd}}), \qquad \text{(new)}$$

  where α, θ are numerical classes; τ is a stability condition. They are weighted motives for semistable moduli stacks.

- Applying motivic integration gives motivic DT invariants

$$\mathrm{DT}_\alpha(\tau) \in \mathbb{Q},$$
$$\mathrm{DT}^{\mathrm{sd}}_\theta(\tau) \in \mathbb{Q}.$$

Motivic invariants

Example. Counting vector spaces:

- **Type A** (known):

$$\sum_{n=1}^{\infty} q^n \cdot \mathrm{DT}_{\mathsf{A}_{n-1}} = \mathrm{Li}_2(q) \, .$$

- **Type B/C**, counting $\mathrm{O}(2n+1)$- or $\mathrm{Sp}(2n)$-vector spaces:

$$\sum_{n=0}^{\infty} q^n \cdot \mathrm{DT}_{\mathsf{B}_n \text{ or } \mathsf{C}_n} = (1-q)^{-1/4} \, .$$

- **Type D**, counting $\mathrm{O}(2n)$-vector spaces:

$$\sum_{n=0}^{\infty} q^n \cdot \mathrm{DT}_{\mathsf{D}_n} = (1-q)^{1/4} \, .$$

Motivic invariants

Wall-crossing.

When one varies the stability condition τ, invariants are related by wall-crossing, expressed in terms of the Lie bracket $[-,-]$ and the twisted module structure $\heartsuit$:

$$\varepsilon_\theta^{\mathrm{sd}}(\tau') = \sum_{\substack{\theta = (\alpha_1 + \alpha_1^\vee) + \cdots + (\alpha_n + \alpha_n^\vee) + \rho; \\ 0 = i_0 < i_1 < \cdots < i_m = n}} (\text{coeff.}) \cdot$$

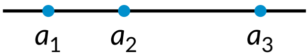
$$\Big[\Big[\dots \Big[\varepsilon_{\alpha_1}(\tau),\ \varepsilon_{\alpha_2}(\tau) \Big], \dots \Big], \ \varepsilon_{\alpha_{i_1}}(\tau) \Big] \ \heartsuit \ \cdots \cdots$$

$$\heartsuit \ \Big[\Big[\dots \Big[\varepsilon_{\alpha_{i_{m-1}+1}}(\tau),\ \varepsilon_{\alpha_{i_{m-1}+2}}(\tau) \Big], \dots \Big], \ \varepsilon_{\alpha_n}(\tau) \Big] \ \heartsuit \ \varepsilon_\rho^{\mathrm{sd}}(\tau) \,.$$

Contents

Vertex algebras and modules

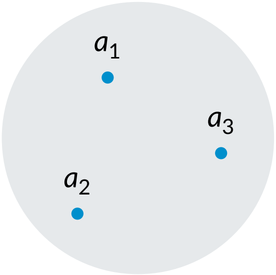
associative algebra



$\Downarrow$

$$a_1 \cdot a_2 \cdot a_3$$

vertex algebra



$\Downarrow$

$$a_1(z_1) \cdot a_2(z_2) \cdot a_3(z_3)$$

Vertex algebras and modules

Vertex algebras.

- The product

  $$a_1(z_1) \cdots a_n(z_n)$$

  is meromorphic in z_i, and only has singularities when $z_i = z_j$ for some $i \neq j$.

- This describes the local structure of conformal field theories in physics.

Vertex algebras and modules

Joyce vertex algebra.

- $\mathcal{M}$: moduli of objects of a $\mathbb{C}$-abelian category.

Then the homology $H_\bullet(\mathcal{M};\mathbb{C})$ is a vertex algebra, defined as follows.

- Let $\oplus : \mathcal{M}^n \to \mathcal{M}$ be the direct sum map.
- Let $Ext_{i,j} \to \mathcal{M}^n$ be the Ext complex, with fibres $\mathrm{Ext}^\bullet(E_i, E_j)$.
- Introduce the notation

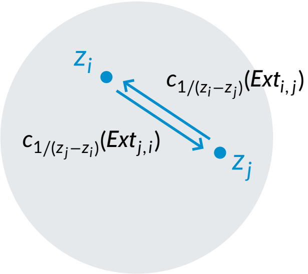
$$c_{1/z}(Ext_{i,j}) = \sum_{n\geq 0} \frac{1}{z^n} c_n(Ext_{i,j})\,.$$

Vertex algebras and modules

Joyce vertex algebra.

The vertex operation is given by

$$a_1(z_1) \cdots a_n(z_n) = \oplus_* \Bigg((a_1 \boxtimes \cdots \boxtimes a_n) \cap$$

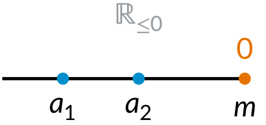


$$\Bigg)$$

where $a_1, \ldots, a_n \in H_\bullet(\mathcal{M})$.

Vertex algebras and modules

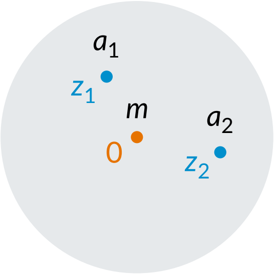
associative algebra module



$\Downarrow$

$$a_1 \cdot a_2 \cdot m$$

vertex algebra module



$\Downarrow$

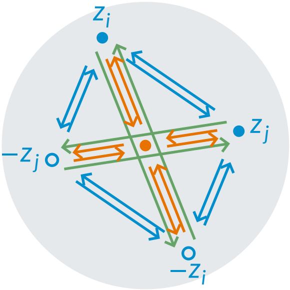
$$a_1(z_1) \cdot a_2(z_2) \cdot m$$

Vertex algebras and modules

Theorem.

The homology $H_\bullet(\mathcal{M}^{\mathrm{sd}})$ is a twisted module for $H_\bullet(\mathcal{M})$:

$$a_1(z_1) \cdots a_n(z_n) \cdot m = \oplus_*^{\mathrm{sd}} \left((a_1 \boxtimes \cdots \boxtimes m) \cap \right)$$



where $a_1, \ldots, a_n \in H_\bullet(\mathcal{M})$ and $m \in H_\bullet(\mathcal{M}^{\mathrm{sd}})$.

Being twisted means allowing extra singularities at $z_i = -z_j$.

Vertex algebras and modules

Lie algebras and twisted modules.

- Every vertex algebra V gives rise to a Lie algebra $V/T(V)$:

$$[a, b] = \operatorname{Res}_{z_1=z_2} a(z_1)\, b(z_2)\,.$$

- Every twisted module M for V gives rise to a twisted module for the Lie algebra $V/T(V)$:

$$a \heartsuit m = \operatorname{Res}_{z=0} a(z)\, m\,,$$

  satisfying the four-term Jacobi identity.

- These coincide with structures in the motivic setting.

Contents

Homological invariants

In type A.

- Joyce (2021) constructed enumerative invariants

$$[\mathcal{M}^{\mathrm{ss}}_{\alpha}(\tau)]_{\mathrm{inv}} \in V/T(V) \simeq H_{\bullet}(\mathcal{M}^{\mathrm{pl}})\,.$$

  Here, $\mathcal{M}^{\mathrm{pl}}$ is the $\mathbb{G}_{\mathrm{m}}$-rigidification of $\mathcal{M}$.

- These generalize fundamental classes, in that

$$[\mathcal{M}^{\mathrm{ss}}_{\alpha}(\tau)]_{\mathrm{inv}} = [\mathcal{M}^{\mathrm{ss}}_{\alpha}(\tau)]_{\mathrm{fund}}$$

  when the latter exists.

- They satisfy wall-crossing formulae when varying the stability condition τ, written in terms of the Lie bracket $[-,-]$ on $V/T(V)$.

Homological invariants

In type B/C/D.

- We conjecture the existence of enumerative invariants

$$[\mathcal{M}_{\theta}^{\mathrm{sd,ss}}(\tau)]_{\mathrm{inv}} \in H_{\bullet}(\mathcal{M}^{\mathrm{sd}}) .$$

- We should have

$$[\mathcal{M}_{\theta}^{\mathrm{sd,ss}}(\tau)]_{\mathrm{inv}} = [\mathcal{M}_{\theta}^{\mathrm{sd,ss}}(\tau)]_{\mathrm{fund}}$$

  when the latter exists.

- They should satisfy wall-crossing formulae when varying the stability condition τ, written in terms of the Lie bracket $[-,-]$ and the twisted module structure $\heartsuit$.

Homological invariants

Wall-crossing.

We should have the wall-crossing formulae

$$[\mathcal{M}_\theta^{\mathrm{sd,ss}}(\tau')]_{\mathrm{inv}} = \sum_{\substack{\theta = (\alpha_1 + \alpha_1^\vee) + \cdots + (\alpha_n + \alpha_n^\vee) + \rho; \\ 0 = i_0 < i_1 < \cdots < i_m = n}} (\text{coeff.}) \cdot$$

$$\Big[\ldots \Big[[\mathcal{M}_{\alpha_1}^{\mathrm{ss}}(\tau)]_{\mathrm{inv}}, \ldots \Big], [\mathcal{M}_{\alpha_{i_1}}^{\mathrm{ss}}(\tau)]_{\mathrm{inv}} \Big] \heartsuit \cdots \cdots$$
$$\heartsuit \Big[\ldots \Big[[\mathcal{M}_{\alpha_{i_{m-1}+1}}^{\mathrm{ss}}(\tau)]_{\mathrm{inv}}, \ldots \Big], [\mathcal{M}_{\alpha_n}^{\mathrm{ss}}(\tau)]_{\mathrm{inv}} \Big] \heartsuit [\mathcal{M}_\rho^{\mathrm{sd,ss}}(\tau)]_{\mathrm{inv}}$$

with precisely the same coefficients as in the motivic case.

Homological invariants

Theorem.

The conjectured invariants

$$[\mathcal{M}_{\theta}^{\mathrm{sd,ss}}(\tau)]_{\mathrm{inv}}$$

exist for self-dual quivers with no oriented loops. They satisfy the above wall-crossing formulae.

However, we can only prove that

$$[\mathcal{M}_{\theta}^{\mathrm{sd,ss}}(\tau)]_{\mathrm{inv}} = [\mathcal{M}_{\theta}^{\mathrm{sd,ss}}(\tau)]_{\mathrm{fund}}$$

for small dimension vectors θ, but not for all of them yet.

Contents

Future directions

- Construct homological invariants $[\mathcal{M}_\theta^{\mathrm{sd,ss}}(\tau)]_{\mathrm{inv}}$ in type B/C/D, possibly via some notion of stable pairs.
- Construct DT invariants for Calabi–Yau 3-folds in type B/C/D.
- Construct the BPS vector space in type B/C/D, analogous to the BPS Lie algebra in type A.
- Find the geometric or physical meaning of the Joyce vertex algebra and the twisted module.
- Generalize this theory to arbitrary reductive groups.

Thank you!