Counting sheaves on curves

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Overview

The setup.

- X — a smooth projective curve over $\mathbb{C}$, i.e. a compact Riemann surface.
- $\mathcal{M}$ — moduli stack of coherent sheaves on X. (Roughly, moduli space of holomorphic vector bundles.)

In physics.

- $\mathcal{M}$ is, roughly, the moduli space of vacua in 2d Yang–Mills theory on X, with gauge group $\mathrm{SU}(n)$.
- The partition function of the theory provides rich geometric information about $\mathcal{M}$. (Witten 1992)

Overview

Question.

- Study the geometry of $\mathcal{M}$ — homology, intersection pairings, volume, …

$$\Uparrow$$

- Homological counting invariants of $\mathcal{M}$ — roughly, the fundamental class "$[\mathcal{M}]_{\text{fund}}$".

Problems.

- $\mathcal{M}$ can be non-Hausdorff.
- $\mathcal{M}$ can be singular.

Overview

Classical approach.

- Use geometric invariant theory to remove non-Hausdorff points. This gives a nice-behaving semistable locus

$$\mathcal{M}^{\mathrm{ss}} \subset \mathcal{M},$$

  without losing much information.

- When r and d are coprime, the connected component

$$\mathcal{M}^{\mathrm{ss}}_{(r,d)} \subset \mathcal{M}^{\mathrm{ss}}$$

  consisting of rank r, degree d vector bundles is smooth.

Overview

Previous work.

- Witten (1992) gave physical predictions of intersection pairings
$$\int_{\mathcal{M}^{\mathrm{ss}}_{(r,d)}} \alpha \cdot \beta$$
when r and d are coprime, for $\alpha, \beta \in H^*(\mathcal{M}^{\mathrm{ss}}_{(r,d)}; \mathbb{Q})$.
- Jeffrey–Kirwan (1998) proved Witten's formulae, using equivariant localization.

Overview

New approach.

- Joyce (2021) defined invariants $[\mathcal{M}_\alpha^{\mathrm{ss}}]_{\mathrm{inv}} \in H_*(\mathcal{M};\mathbb{Q})$ in a very general setting, using stable pairs. Here $\alpha = (r, d)$.
- When r and d are coprime, $[\mathcal{M}_{(r,d)}^{\mathrm{ss}}]_{\mathrm{inv}} = [\mathcal{M}_{(r,d)}^{\mathrm{ss}}]_{\mathrm{fund}}$.
- Joyce proved wall-crossing formulae for his invariants, expressed in terms of a vertex algebra structure on $H_*(\mathcal{M};\mathbb{Q})$.
- Using this machinery, I obtained explicit formulae for the classes $[\mathcal{M}_{(r,d)}^{\mathrm{ss}}]_{\mathrm{inv}}$, in terms of a regularized sum — a divergent infinite sum.

Overview

Main Theorem.

$$[\mathcal{M}^{\mathrm{ss}}_{(r,d)}]_{\mathrm{inv}} = \sum_{\substack{(d_1, \ldots, d_r) \in \text{some lattice} \\ d_1 + \cdots + d_r = d}} (\text{coeff.}) \cdot \Big[\Big[\cdots \Big[[\mathcal{M}^{\mathrm{ss}}_{(1,d_1)}]_{\mathrm{fund}} , [\mathcal{M}^{\mathrm{ss}}_{(1,d_2)}]_{\mathrm{fund}}\Big], \ldots \Big], [\mathcal{M}^{\mathrm{ss}}_{(1,d_r)}]_{\mathrm{fund}}\Big].$$

Remarks.

- $\mathcal{M}^{\mathrm{ss}}_{(1,d_i)} \simeq J(X)$, the Jacobian variety of X, which is a torus T^{2g}.
- The Lie brackets come from Joyce's vertex algebra.

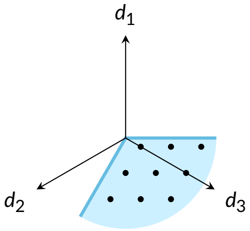
Overview

Examples.

- When $g = 2$, the volume $\mathrm{vol}(\mathcal{M}^{\mathrm{ss,fd}}_{(2,1)})$ is

$$1 + 3 + 5 + 7 + \cdots = \frac{1}{12}.$$

- When $(r, d) = (3, 1)$, we are summing over this lattice:



Overview

Remarks.

- When r and d are coprime, my formulae recover classical results on intersection pairings on $\mathcal{M}^{\mathrm{ss}}_{(r,d)}$.
- Witten expressed his formulae as a convergent sum over a lattice, which is related to my lattice by something similar to a Fourier transform.
- When r and d are not coprime, I believe my results produce physically expected numbers. (The case $r = 2$ is in the literature.)

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Moduli of sheaves

Automorphism groups.

- Moduli stacks care about automorphism groups of objects.
- For $E \in \mathrm{Coh}(X)$, we always get the scalar automorphisms

$$\mathbb{G}_{\mathrm{m}} \simeq \{\lambda \cdot \mathrm{id} \mid \lambda \in \mathbb{C}^\times\} \subset \mathrm{Aut}(E).$$

- We can remove them to get something closer to the classical moduli space. In other words, we consider

$$\mathrm{Aut}^{\mathrm{pl}}(E) = \mathrm{Aut}(E)/\mathbb{G}_{\mathrm{m}}\,,$$

where pl = projective linear.

Moduli of sheaves

Variants of moduli stacks.

- $\mathcal{M}$ – moduli stack of coherent sheaves on X.
- $\mathcal{M}^{\mathrm{pl}} = \mathcal{M}/[*/\mathbb{G}_{\mathrm{m}}]$, the stack $\mathcal{M}$ with scalar automorphisms removed.
- $\mathcal{M}^{\mathrm{ss}}_{(r,d)} \subset \mathcal{M}^{\mathrm{pl}}$ consists of semistable coherent sheaves of rank r, degree d.

  When $r > 0$, all semistable sheaves are vector bundles.
- $\mathcal{M}^{\mathrm{ss,fd}}_{(r,d)} \subset \mathcal{M}^{\mathrm{ss}}_{(r,d)}$ consists of sheaves with a fixed (chosen) determinant line bundle (when $r > 0$).

Moduli of sheaves

Homology.

- The rational cohomology of $\mathcal{M}$ is a free supercommutative polynomial algebra:

$$H^*(\mathcal{M};\mathbb{Q}) \simeq \mathbb{Q}[S_{\alpha;j,k,\ell}],$$

  with $\alpha \in K(X)$, $1 \le j \le b^k(X)$ (the k-th Betti number of X), and $\ell > k/2$, with $\deg S_{\alpha;j,k,\ell} = 2\ell - k$.

- By duality, we can define an isomorphism of vector spaces

$$H_*(\mathcal{M};\mathbb{Q}) \simeq \mathbb{Q}[s_{\alpha;j,k,\ell}],$$

  with $\deg s_{\alpha;j,k,\ell} = 2\ell - k$.

Moduli of sheaves

Vertex algebras.

- Joyce proved that the graded vector space

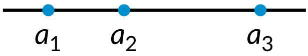
$$H_*(\mathcal{M};\mathbb{Q}) \simeq \mathbb{Q}[s_{\alpha;j,k,\ell}]$$

  has the structure of a vertex algebra.

- This is very surprising. Vertex algebras are very complicated algebraic structures, arising from 2d conformal field theory in physics. But this structure exists in a very general setting, which seems to have nothing to do with two dimensions.

Moduli of sheaves

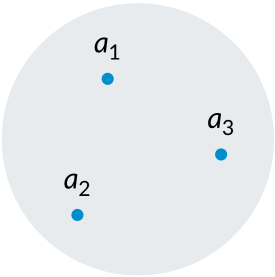
associative algebra



$\Downarrow$

$$a_1 \cdot a_2 \cdot a_3$$

vertex algebra



$\Downarrow$

$$a_1(z_1) \cdot a_2(z_2) \cdot a_3(z_3)$$

Moduli of sheaves

- In a vertex algebra $\mathcal{V}$, elements $a \in \mathcal{V}$ are thought of as fields $a(z)$ on the formal disk, which are meromorphic and operator-valued:

$$a \in \mathcal{V} \quad \Longrightarrow \quad a(z) \in \mathrm{End}(\mathcal{V})\,[[z, z^{-1}]]$$

- The product $a(z)\,b$ is often denoted by

$$Y(a, z)\,b = \sum_n a_n(b) \cdot z^{-n-1} \qquad \in \mathcal{V}[[z]]\,[z^{-1}],$$

where $a_n(b) \in \mathcal{V}$.

Moduli of sheaves

- Taking the residue

$$[a,b] = a_0(b) = \oint_0 a(z)\, b\, dz$$

associates a Lie algebra to every vertex algebra, which is a quotient of the vertex algebra.

- In our setting,

vertex algebra		Lie algebra
$H_*(\mathcal{M};\mathbb{Q})$	$\Longrightarrow$	$H_*(\mathcal{M}^{\mathrm{pl}};\mathbb{Q})$

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Homological invariants

Pairs.

Fix, once and for all, a line bundle $L \to X$. We often take $L = \mathcal{O}(-N)$ for $N \gg 0$.

- A pair is a map
$$\rho \colon L \otimes V \to E,$$
with V a vector space and $E \in \mathrm{Coh}(X)$.
- We are mostly concerned with the case $V = \mathbb{C}$. Then a pair is just a sheaf E with a section of $E \otimes L^{-1}$.

Homological invariants

Moduli of pairs.

- $\acute{\mathcal{M}}_{(r,d),e}$ – moduli stack of pairs, where the sheaf E is of class (r, d) and the vector space V is of dimension e.
- $\acute{\mathcal{M}}^{\mathrm{pl}}_{(r,d),e} = \acute{\mathcal{M}}_{(r,d),e}/[*/\mathbb{G}_{\mathrm{m}}]$, with scalar automorphisms removed.
- $\acute{\mathcal{M}}^{\mathrm{ss}}_{(r,d),e} \subset \acute{\mathcal{M}}^{\mathrm{pl}}_{(r,d),e}$ – the semistable locus, for a certain stability condition.
- $\acute{\mathcal{M}}^{\mathrm{ss}}_{(r,d),1}$ is well-behaved, since stable = semistable. We have the Behrend–Fantechi virtual fundamental class

$$[\acute{\mathcal{M}}^{\mathrm{ss}}_{(r,d),1}]_{\mathrm{virt}} \in H_*(\acute{\mathcal{M}}^{\mathrm{pl}}; \mathbb{Z}).$$

Homological invariants

Joyce's invariants.

- The forgetful map
$$\Pi : \acute{\mathcal{M}}^{\mathrm{pl}}_{(r,d),1} \longrightarrow \mathcal{M}^{\mathrm{pl}}_{(r,d)}$$
sending a pair to its underlying sheaf, looks like a projective bundle. So we can define the invariants as a "virtual pushforward"
$$[\mathcal{M}^{\mathrm{ss}}_{(r,d)}]_{\mathrm{inv}} = \Pi_! \, [\acute{\mathcal{M}}^{\mathrm{ss}}_{(r,d),1}]_{\mathrm{virt}} \in H_*(\mathcal{M}^{\mathrm{pl}}; \mathbb{Q}).$$
- Recall that $H_*(\mathcal{M}^{\mathrm{pl}}; \mathbb{Q})$ has a Lie algebra structure, inherited from the vertex algebra structure on $H_*(\mathcal{M}; \mathbb{Q})$.

Homological invariants

Wall-crossing.

There are different interesting stability conditions for pairs on curves. Their invariants are related by wall-crossing formulae. For example, one of them reads

$$[\acute{\mathcal{M}}^{\mathrm{ss}}_{(r,d),1}]_{\mathrm{virt}} = \sum_{\substack{(r,d) = (r_1,d_1) + \cdots + (r_m,d_m), \\ m \geq 1,\ r_i > 0,\ d_i/r_i = d/r \text{ for all } i}} (\text{coeff.}) \cdot$$

$$\Big[\Big[\dots \Big[\Big[e^{((0,0),1)},\ [\mathcal{M}^{\mathrm{ss}}_{(r_1,d_1)}]_{\mathrm{inv}} \Big],\ [\mathcal{M}^{\mathrm{ss}}_{(r_2,d_2)}]_{\mathrm{inv}} \Big],\ \dots \Big],\ [\mathcal{M}^{\mathrm{ss}}_{(r_m,d_m)}]_{\mathrm{inv}} \Big].$$

This particular one is an analogue of what is known as DT–PT wall-crossing.

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The regularized sum

Idea.

For any expression a (e.g. $a = 2$), define

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$

Example.

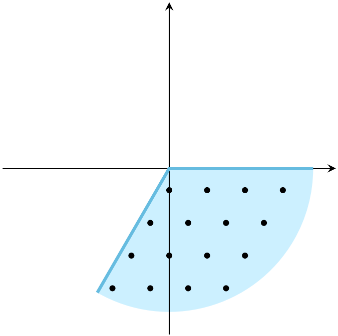
$$\sum_{n=0}^{\infty} \frac{e^{nz}}{z^2} = \frac{1}{z^2(1-e^z)}$$

$$\Big\Downarrow \ \mathrm{res}_{z=0}$$

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

The regularized sum

In general, the regularized sum is defined for geometric series on a lattice, such as this one:

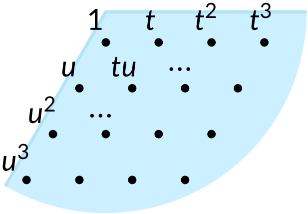


The regularized sum

Definition.

The definition is best given visually:

$$\sum$$

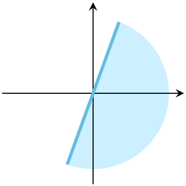


$$= \frac{1}{(1-t)(1-u)} \,.$$

The regularized sum

Important property.

Half-planes are always zero:

$$\sum$$  $$= \quad 0\,.$$

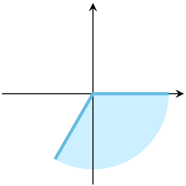
Reason:

$$\cdots + t^{-2} + t^{-1} + 1 + t + t^2 + \cdots = \frac{t^{-1}}{1 - t^{-1}} + \frac{1}{1 - t} = 0.$$

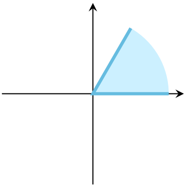
The regularized sum

This gives us relations such as:

$\sum$



$= -\sum$



These relations define a quotient space of the vector space spanned by polyhedral cones, where some very interesting combinatorics takes place.

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Main result

Main Theorem.

For any $(r, d) \in \mathbb{Z}_{>0} \times \mathbb{Z}$, we have

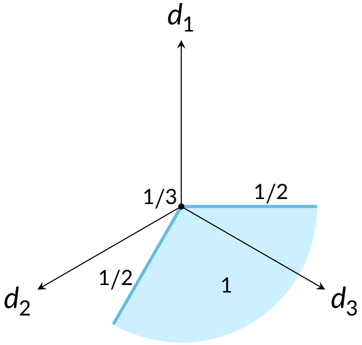
$$[\mathcal{M}^{\mathrm{ss}}_{(r,d)}]_{\mathrm{inv}} = \sum_{\substack{d = d_1 + \cdots + d_r \ \ (d_i \in \mathbb{Z}) \\ \frac{d_1 + \cdots + d_i}{i} \leq \frac{d}{r}, \ i = 1, \ldots, r-1}} \frac{1}{m+1} \cdot$$

$$\Big[\Big[\cdots \Big[[\mathcal{M}^{\mathrm{ss}}_{(1,d_1)}]_{\mathrm{fund}} \, , \ [\mathcal{M}^{\mathrm{ss}}_{(1,d_2)}]_{\mathrm{fund}} \Big], \ \ldots \Big], \ [\mathcal{M}^{\mathrm{ss}}_{(1,d_r)}]_{\mathrm{fund}} \Big],$$

where m is the number of equalities in the summation condition.

Main result

When $r = 3$, the summation region is shown below, in the plane $d_1 + d_2 + d_3 = d$. The small numbers indicate the coefficients.



Main result

Remarks.

- When r and d are coprime, no integral points lie on the boundary of the region, so all coefficients are 1.

  This case corresponds to classical results of Witten and Jeffrey–Kirwan.

- The fixed determinant invariants $[\mathcal{M}^{\mathrm{ss,fd}}_{(r,d)}]_{\mathrm{inv}}$ can be easily obtained from $[\mathcal{M}^{\mathrm{ss}}_{(r,d)}]_{\mathrm{inv}}$.

- One can write down explicit formulae (very long) for the invariants $[\mathcal{M}^{\mathrm{ss}}_{(r,d)}]_{\mathrm{inv}}$ and $[\mathcal{M}^{\mathrm{ss,fd}}_{(r,d)}]_{\mathrm{inv}}$, in terms of the generators $s_{j,k,\ell} \in H_*(\mathcal{M}^{\mathrm{pl}}; \mathbb{Q})$.

Main result

The formula.

$$[\mathcal{M}^{ss}_{(r,d)}]_{\mathrm{inv}} = \mathrm{res}_{z_{r-1}} \circ \cdots \circ \mathrm{res}_{z_1} \Bigg\{ \frac{(-1)^{(g-1)r(r-1)/2+(r-1)(d-1)}}{r \cdot \prod_{0 \le i < j \le r-1} (z_i - z_j)^{2g-2}} \cdot$$

$$\sum_{\substack{0 \le m \le \gcd(r,d)-1 \\ 1 \le i_1 < \cdots < i_m \le r-1 \\ \text{such that } i_k d/r \in \mathbb{Z} \text{ for all } k}} \frac{(-1)^m}{m+1} \cdot \frac{1}{\prod_{\substack{1 \le i \le r-1 \\ i \ne i_k \text{ for any } k}} \left[1 - \exp\left(\sum_{\ell=1}^{\infty} \frac{\tilde{z}_i^{\ell} - \tilde{z}_{i-1}^{\ell}}{\ell!} s_{1,2,\ell+1} \right) \right]} \cdot$$

$$\prod_{i=0}^{r-1} \left[\exp\left(\tilde{z}_i \left(s_{1,0,1} + \left(\left\lfloor \frac{(i+1)d}{r} \right\rfloor - \left\lfloor \frac{id}{r} \right\rfloor \right) s_{1,2,2} + \sum_{j,k,\ell} s_{j,k,\ell+1} \frac{\partial}{\partial s_{j,k,\ell}} \right) \right) \cdot \right.$$

$$\left. \prod_{j=1}^{g} (s_{j,1,1} s_{j+g,1,1} - s_{1,2,2}) \right] \Bigg\} \Bigg|_{z_0=0}$$

where $\tilde{z}_i = z_i - (z_0 + \cdots + z_{r-1})/r$.

Main result

Examples.

- If we compute the volume $\mathrm{vol}(\mathcal{M}^{\mathrm{ss,fd}}_{(2,1)})$, when $g = 1$, we get

$$0 + 0 + 0 + 0 + \cdots = 1.$$

  When $g = 2$, we get

$$1 + 3 + 5 + 7 + \cdots = \frac{1}{12}.$$

- When $g = 1$, the fixed determinant moduli stack has virtual dimension 0, and

$$[\mathcal{M}^{\mathrm{ss,fd}}_{(r,d)}]_{\mathrm{inv}} = (-1)^{(r-1)(d-1)}.$$

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A taste of the proof

The proof is by induction on rank.

Ingredient 1.

Wall-crossing between two stability conditions for pairs gives

$$[\acute{\mathcal{M}}^{ss}_{(r,d),1}]_{virt} = \sum_{\substack{(r,d) = (r_0, d_0) + \cdots + (r_m, d_m), \\ m \geq 1,\ r_i > 0 \text{ for all } i, \\ \text{such that } d_0/r_0 < d_1/r_1 \leq \cdots \leq d_m/r_m}} (\text{coeff.}) \cdot \Big[\Big[\cdots \Big[[\acute{\mathcal{M}}^{ss}_{(r_0,d_0),1}]_{virt} , [\mathcal{M}^{ss}_{(r_1,d_1)}]_{inv} \Big], \ldots \Big], [\mathcal{M}^{ss}_{(r_m,d_m)}]_{inv} \Big].$$

This is a finite sum. The r.h.s. only involves invariants of rank $< r$.

A taste of the proof

Ingredient 1 (continued).

Assume that the main theorem is true for rank $< r$. Plugging in the regularized sum expression, we obtain

$$[\acute{\mathcal{M}}^{\mathrm{ss}}_{(r,d),1}]_{\mathrm{virt}} = \sum_{(d_1,\dots,d_r)\in\Lambda} (\text{coeff.}) \cdot$$

$$\Big[\Big[\cdots\Big[[\acute{\mathcal{M}}^{\mathrm{ss}}_{(1,d_1),1}]_{\mathrm{virt}}\,,\ [\mathcal{M}^{\mathrm{ss}}_{(1,d_2)}]_{\mathrm{fund}}\Big],\ \dots\Big],\ [\mathcal{M}^{\mathrm{ss}}_{(1,d_r)}]_{\mathrm{fund}}\Big].$$

This effectively computes all the pair invariants, as the rank 1 invariants are simple.

A taste of the proof

Ingredient 2.

The analogue of DT–PT wall-crossing gives

$$[\acute{\mathcal{M}}^{\mathrm{ss}}_{(r,d),1}]_{\mathrm{virt}} = \sum_{\substack{(r,d) = (r_1,d_1) + \cdots + (r_m,d_m), \\ m \geq 1,\ r_i > 0,\ d_i/r_i = d/r \text{ for all } i}} (\text{coeff.}) \cdot$$

$$\Big[\Big[\cdots\Big[\Big[e^{((0,0),1)},\ [\mathcal{M}^{\mathrm{ss}}_{(r_1,d_1)}]_{\mathrm{inv}}\Big],\ [\mathcal{M}^{\mathrm{ss}}_{(r_2,d_2)}]_{\mathrm{inv}}\Big],\ \cdots\Big],\ [\mathcal{M}^{\mathrm{ss}}_{(r_m,d_m)}]_{\mathrm{inv}}\Big].$$

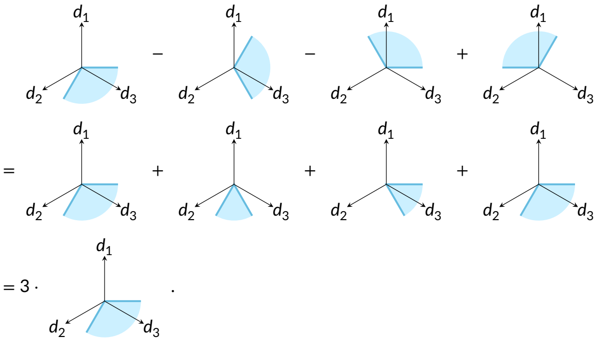
where the sum is a finite sum.

The l.h.s. is known. The only unknown on the r.h.s. is $[\mathcal{M}^{\mathrm{ss}}_{(r,d)}]_{\mathrm{inv}}$. It remains to check that the expression for $[\mathcal{M}^{\mathrm{ss}}_{(r,d)}]_{\mathrm{inv}}$ is consistent with this wall-crossing.

A taste of the proof

Ingredient 3.

What we are left to do is pure combinatorics. In rank 3, this is illustrated below.



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Future directions

- Find a geometric or physical interpretation of the regularized sum formula.
- Generalization to principal bundles.
- Generalization to higher dimensional varieties (Related: Joyce's ongoing project on surfaces).
- Categorification of the regularized sum.

Thank you!